A MEAN FIELD MODEL FOR THE INTERACTIONS BETWEEN FIRMS ON THE MARKETS OF THEIR INPUTS

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Abstract. We consider an economy made of competing firms which are heterogeneous in their capital and use several inputs for producing goods. Their consumption policy is fixed rationally by maximizing a utility and their capital cannot fall below a given threshold (state constraint). We aim at modeling the interactions between firms on the markets of the different inputs on the long term. The stationary equlibria are described by a system of coupled non-linear differential equations: a Hamilton-Jacobi equation describing the optimal control problem of a single atomistic firm; a continuity equation describing the distribution of the individual state variable (the capital) in the population of firms; the equilibria on the markets of the production factors. We prove the existence of equilibria under suitable assumptions.

1. Introduction. We consider an economy made of competing firms which are heterogeneous in their capital, and use several inputs for producing goods. These inputs, or factors of production, may include raw materials, energy, manpower, rented surface, etc... We aim at modeling the interactions of the firms on the markets of the different inputs on the long term. We make the following general assumptions:

- the economy is reduced to one sector of activity with a large number (in fact a continuum) of firms competing on the markets of inputs
- the firms choose which amount of their capital is invested into production and which amount is consumed (for retributing the owners). Their consumption policy is fixed rationally by maximizing a utility
- the firms are identical in the sense that (1) two different firms with the same capital and quantities of inputs produce the same amounts of goods (2) they have the same utility function
- there is a state constraint: the capital of any firm must not fall below a given threshold, fixed to 0 in the whole paper
- for a given firm, all the others are indistinguishable and the firms interact only via the prices of the different inputs
- a single firm has a negligible impact on the markets
- equilibrium on the markets is reached when supply matches aggregate demand. Supply is assumed to be a given function of prices.
- closure and creation of firms may happen. This will be modeled in what follows.

Because we are interested in long term tendencies, we aim at finding stationary equilibria. The outputs of our model will be

- the distribution of capital
- the optimal investment/consumption policy of the firms given their capital
- the unit prices of the different inputs

Our model falls into the wide class of mean field games. The theory of mean field

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games (*MFGs* for short), has been introduced and studied in the pioneering works of J-M. Lasry and P-L. Lions [16, 13, 14, 15], and aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of agents tends to infinity. It supposes that the rational agents are indistinguishable and individually have a negligible influence on the game, and that each individual strategy is influenced by some averages of quantities depending on the states (or the controls) of the other agents. The applications of MFGs are numerous, from economics to the study of crowd motion. For useful reference on mean field games, one can see for example [11, 9, 2, 1].

Our model will be summarized by a system of coupled non-linear differential equations: (1) a Hamilton-Jacobi-Bellman equation describing the optimal control problem of a single atomistic firm; (2) a continuity equation describing the distribution of the individual state variable (the capital) in the population of firms; (3) the equilibria on the markets of the production factors.

The present model has some similarities with the time continuous Aiyagari-Bewley-Huggett models [7, 4, 12] studied in [1, 3]. In particular, they all lead to a better understanding of the individual accumulation of capital/investment policy. In the present paper, a key aspect for proving the existence of equilibria is the regularity properties of the individual optimal policies.

The paper is organized as follows: the model, the main results and important examples are presented in Section 2. The mathematical results concerning the optimal control problem of a single firm given the prices of inputs are proved in Section 3. As already mentioned, the stress will be put on regularity properties of the solutions, which will play an important role in the remaining part of the paper. Then, the distribution of capital among the firms given the prices of inputs will be studied in Section 4: in particular, we will prove that under the assumptions made, the distribution is absolutely continuous with respect to Lebesgue measure. Finally, in Section 5, we use Brouwer topological degree in order to obtain the existence of equilibria.

For keeping the length of the paper reasonable, we have chosen not to discuss the numerical simulations that we have carried out for a model with two factors of production: the manpower and the surface rented by the firms. We refer to [17] for a description of these simulations, a discussion of the results and comparisons with available statistics.

2. The model and the main results. In what follow, we give more details and write down the different equations which summarize our model. First, in paragraph 2.1, we address the strategy of a single firm given the prices of the inputs. Second, in paragraph 2.2, we propose a model for the distribution of capital, supposing again that the prices of the inputs are given. From the two steps above, we can deduce the aggregate demand for the different production factors. Finally, the model is closed by matching the aggregate demand with the exogenous supply of production factors. In the three steps mentioned above, we make some assumptions which allow us to prove the existence of a mean field equilibrium. In subsection 2.4 below, we give examples in which these assumptions are satisfied.

In the following, we set $\mathbb{R}_+ = [0, +\infty)$.

2.1. The optimal control problem of a single firm given the prices of inputs. The output of a given firm is $F(k, \ell) \in \mathbb{R}_+$, where $k \in \mathbb{R}_+$ and $\ell \in \mathbb{R}_+^d$ respectively stand for the capital of the firm and for the quantities of the different inputs it uses. The function $F : \mathbb{R}_+ \times \mathbb{R}_+^d \to \mathbb{R}_+$ is the production function.

Let $w \in (0,+\infty)^d$ be the collection of the unit prices of the different factors of production: depending on $i \in \{1,\dots,d\}$, w_i may stand for the unit price of a raw material, the annual salary of a class of workers, the rental price of a unit of surface. The benefits of the firm in a unit of time are therefore given by $F(k,\ell) - w\cdot\ell - \delta k$, where $\delta \geqslant 0$ is the rate of depreciation of capital.

The dynamics of the capital of a given firm is described by

$$\frac{dk}{dt}(t) = F(k(t),\ell(t)) - w\cdot\ell(t) - \delta k(t) - c(t), \tag{2.1}$$

where $c(t)$ stands for the consumption at time t, for example the share of the benefits that goes to the owners of the firm. The firm has two variables of control, its consumption $c(t) \in \mathbb{R}_+$ and the quantities of inputs $\ell(t) \in \mathbb{R}_+^d$.

The firms face the problem of how to split their benefits between consumption and investments that produce growth. A given firm determines its policy by maximizing the following payoff:

$$\int_0^{+\infty} U(c(t))e^{-\rho t}dt, \tag{2.2}$$

where $U : [0,+\infty) \to [-\infty,+\infty)$ is a utility function and ρ is a positive discount factor.

It aims at finding the controls $t \mapsto c(t) \in [0,+\infty)$ and $t \mapsto \ell(t) \in [0,+\infty)^d$ which maximize (2.2), under the constraint that its capital stay nonnegative (state constraint). The value of the optimal control problem when the firm has a capital $k_0 \geqslant 0$ is

$$\begin{aligned} u(k_0,w) \quad &= \quad \sup_{c,\,\ell,\,k} \int_0^{+\infty} U(c(t))e^{-\rho t}dt \\ & \qquad \text{subject to} \\ & \qquad \begin{cases} c \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+), \quad \ell \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+^d), \quad k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+), \\ k \text{ satisfies (2.1) for a.a. } t > 0, \\ k(0) = k_0, \\ k(t) \geqslant 0 \text{ for all } t. \end{cases} \end{aligned} \tag{2.3}$$

We will see that under suitable assumptions, namely Assumptions 2.1 and 2.2 below, $u(k_0,w) \in \mathbb{R}$ for all $k_0 \in (0,+\infty)$.

We expect that the value function u can be found by solving a Hamilton-Jacobi equation in $(0,+\infty)$ with state constraints at $k=0$ (from the dynamic programming principle). Let the Hamiltonian $H : \mathbb{R}_+ \times \mathbb{R} \times (0,+\infty)^d \to (-\infty,+\infty]$ be defined as follows: for all $k \in \mathbb{R}_+$ and $q \in \mathbb{R}$,

$$H(k,q,w) = \sup_{c\in\mathbb{R}_+,\,\ell\in\mathbb{R}_+^d} \{U(c) + q\,(F(k,\ell) - w\cdot\ell - \delta k - c)\} \tag{2.4}$$

$$= \sup_{c\in\mathbb{R}_+} \{U(c) - cq\} + f(k,w)q, \tag{2.5}$$

where $f : \mathbb{R}_+ \times (0,+\infty)^d \to \mathbb{R}$ is the *net output* function:

$$f(k,w) = \sup_{\ell\in\mathbb{R}_+^d} \{F(k,\ell) - w\cdot\ell\} - \delta k. \tag{2.6}$$

REMARK 2.1. *By contrast with simpler applications of mean field games to price formation, see e.g. [10], the Hamiltonian of the problem does not involve a quantity which depends separately/additively on the price vector w and on q.*

The Hamilton-Jacobi equation then reads:

$$-\rho u(k,w) + H\left(k, \frac{\partial u}{\partial k}(k,w), w\right) = 0, \qquad \text{in } (0,+\infty). \tag{2.7}$$

Recall that a function $\psi : \mathbb{R}_+^m \to \mathbb{R}$ is monotone if and only if for every $z, \tilde{z} \in \mathbb{R}_+^m$,

$$z \leqslant \tilde{z} \qquad \Rightarrow \qquad \psi(z) \leqslant \psi(\tilde{z}),$$

where the partial order $\leqslant$ on $\mathbb{R}^m$ is defined as follows:

$$z \leqslant \tilde{z} \quad \text{if and only if} \quad z_i \leqslant \tilde{z}_i, \;\; \forall i \in \{1,\dots,m\}..$$

We make the following assumptions on U and F:

ASSUMPTION 2.1 (Assumptions on U). *The utility $U : \mathbb{R}_+ \to [-\infty,+\infty)$ has the following properties:*
i) U is C^2 on $(0,+\infty)$.
ii) U is increasing and strictly concave in $(0,+\infty)$.
iii) $\lim_{c\to 0^+} U'(c) = +\infty$ and $\lim_{c\to+\infty} U'(c) = 0$.

ASSUMPTION 2.2 (Assumptions on F). *The function F is concave and monotone. For any vector $w \in (0,+\infty)^d$ and for any $k \in \mathbb{R}_+$, the net output $f(k,w)$ defined by (2.6) is finite and achieved by a unique $\ell = \ell^*(k,w) \in \mathbb{R}_+^d$, and ℓ^* is a C^1 function defined on $(0,+\infty)\times(0,+\infty)^d$.*
Moreover,
1. *The function f belongs to $C^0(\mathbb{R}_+ \times (0,+\infty)^d) \cap C^1((0,+\infty)^{d+1})$*
2. *for all $w \in (0,+\infty)^d$, $f(\cdot,w) : \mathbb{R}_+ \to \mathbb{R}$ has the following properties:*
   i) $f(\cdot,w)$ is locally of class $C^{1,1}$ on $(0,+\infty)$
   ii) $f(0,w) \geqslant 0$, $k \mapsto f(k,w)$ is strictly concave and $\lim_{k\to 0^+} \frac{\partial f}{\partial k}(k,w) = +\infty$
   iii) $\lim_{k\to+\infty} \frac{\partial f}{\partial k}(k,w) = -\delta$

REMARK 2.2.
- *From point 2.ii) in Assumption 2.2, $f(\cdot,w)$ is strictly concave. Hence, $\frac{\partial f}{\partial k}(k,w)$ has a limit as $k \to +\infty$, which belongs to $[-\infty,+\infty)$. Therefore, point 2.iii) in Assumption 2.2 is meaningful.*
- *If $\delta = 0$, then the strict concavity of $k \mapsto f(k,w)$ implies that it is increasing in $(0,+\infty)$. Then, because $f(0,w) \geqslant 0$, $f(k,w) > 0$ for all $k > 0$ and has a limit as $k \to +\infty$, which belongs to $(0,+\infty]$.*
- *If $\delta > 0$, then $\lim_{k\to+\infty} f(k,w) = -\infty$, and f is negative for k large enough.*

REMARK 2.3. *It is clear that $-f$ is monotone with respect to w. The optimal quantity of the input labeled i is*

$$\ell_i^*(k,w) = -\frac{\partial f}{\partial w_i}(k,w).$$

In Section 3 below, we are going to prove the following theorem:

Theorem 2.1. *Under Assumptions 2.1 and 2.2, for all $w \in (0, +\infty)^d$, there exists a unique classical solution $u(\cdot, w) \in C^1(0, +\infty)$ of (2.7) with the following property: there exists a critical value $\kappa^*(w) > 0$, such that*

$$H_q\left(k, \frac{\partial u}{\partial k}(k, w), w\right) > 0, \quad \text{for } 0 < k < \kappa^*(w), \tag{2.8}$$

$$H_q\left(k, \frac{\partial u}{\partial k}(k, w), w\right) < 0, \quad \text{for } \kappa^*(w) < k < +\infty. \tag{2.9}$$

Here H_q stands for the partial derivative of H with respect to its second argument. Moreover $\kappa^*(w)$ is characterized by the equation

$$\frac{\partial f}{\partial k}(\kappa^*, w) = \rho. \tag{2.10}$$

The function $u(\cdot, w)$ is strictly concave on $(0, +\infty)$ and belongs to $C^2((0, \kappa^*(w)) \cup (\kappa^*(w), +\infty))$.
Furthermore, $u(\cdot, w)$ is the value function of the optimal control problem (2.3), and $H_q(k, \partial_k u(k, w), w)$ is the optimal investment policy of a firm with capital k.

Remark 2.4.

1. *The existence of $\kappa^*(w) > 0$ such that the capital of all firms converges towards $\kappa^*(w)$ is known as the golden rule of investment [5, Chapter 7].*
2. *We will see in Section 4 below that a firm with an initial capital $k_0 \neq \kappa^*(w)$ never reaches this target capital $\kappa^*(w)$.*

The difficulty in the proof of Theorem 2.1 lies in the fact that the Hamiltonian $H(k, q, w)$ is defined only for nonnegative values of q (i.e. $H(k, q, w) = +\infty$ if $q < 0$) and may blow up as $q \to 0_+$. Hence, classical results on viscosity solutions of Hamilton-Jacobi equations for state constrained optimal control problems cannot be applied in a straightforward manner. We will use a different strategy: in particular, in the simplest case in which $\delta = -\lim_{k\to\infty} \frac{\partial f}{\partial k}(k, w) = 0$, our proof of existence is based on the fact that the function $q \mapsto H(k, q, w)$ is strictly convex, strictly decreasing in $(0, q_{\min})$ and strictly increasing in $(q_{\min}, +\infty)$, where $q_{\min} = U'(f(k, w))$, see Lemma 3.2 below. In this case, our strategy consists in solving two ordinary differential equations by means of shooting methods: the first (resp. second) one involves the inverse of the increasing (resp. decreasing) part of $q \mapsto H(k, q, w)$.

Note that a different strategy has been studied in [17]; it was inspired by a method proposed in [18] for studying Ramsey model of optimal growth with non local externalities. It consists in introducing a relaxed lagrangian version of the original optimal control problem, then obtaining compactness properties which lead to the existence of an optimal control and of a solution of the original problem. However, this approach needs an assumption stronger than Assumption 2.2.

2.2. The distribution of capital given the prices of inputs. The distribution of capital corresponding to the optimal investment policy of the firms is a bounded positive measure on $(0, +\infty)$. In our model, its density is characterized by the following continuity equation:

$$\frac{\partial}{\partial k}\left(m(\cdot, w) H_q\left(\cdot, \frac{\partial u}{\partial k}(\cdot, w), w\right)\right) = \eta(\cdot, u(\cdot, w)) - \nu m(\cdot, w), \tag{2.11}$$

which may first be understood in the sense of distributions. The parameter $\nu \geqslant 0$ is the extinction rate of the firms and the source term η stands for the exogenous creation of firms. Note that the latter term depends on the value u. We make the following assumption:

ASSUMPTION 2.3 (Assumptions on ν and η). *We assume that ν is positive ($\nu > 0$), that η is a continuous function on $[0, +\infty) \times \mathbb{R}$, and that there exists a continuous probability density $\hat{\eta} : \mathbb{R}_+ \to \mathbb{R}_+$ with a compact support contained in $(0, +\infty)$ and a positive constant $\hat{c} \geqslant 1$ such that for all $k \geqslant 0$ and $v \in \mathbb{R}$,*

$$\frac{1}{\hat{c}}\hat{\eta}(k) \leqslant \eta(k, v) \leqslant \hat{c}\hat{\eta}(k).$$

Equation (2.11) is supplemented with the condition

$$\nu \int_{\mathbb{R}_+} m(k, w) dk = \int_{\mathbb{R}_+} \eta(k, u(k, w)) dk. \tag{2.12}$$

Since $H_q(k, \frac{\partial u}{\partial k}(k, w), w) > 0$ for small values of k and $H_q(k, \frac{\partial u}{\partial k}(k, w), w) < 0$ for large values of k, see Theorem 2.1, (2.12) is a weak way to say that the flux $m(\cdot, w) H_q(\cdot, \frac{\partial u}{\partial k}(\cdot, w), w)$ vanishes at $k = 0$ and as $k \to +\infty$.

PROPOSITION 2.2. *Under Assumptions 2.1, 2.2 and 2.3, the unique solution of (2.11)-(2.12) is given by*

$$m(k, w) =$$

$$\begin{cases} \dfrac{1}{b(k, w)} \displaystyle\int_0^k \eta(\kappa, u(\kappa, w)) \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz \right) d\kappa, & \text{if} \quad k < \kappa^*(w), \\ -\dfrac{1}{b(k, w)} \displaystyle\int_k^{+\infty} \eta(\kappa, u(\kappa, w)) \exp\left(\int_k^\kappa \frac{\nu}{b(z, w)} dz \right) d\kappa, & \text{if} \quad k > \kappa^*(w), \end{cases} \tag{2.13}$$

where, for brevity, $b(k, w)$ stands for the optimal investment when the capital is k:

$$b(k, w) = H_q\left(k, \frac{\partial u}{\partial k}(k, w), w\right). \tag{2.14}$$

A key step in the proof of Proposition 2.2 consists of showing that the quantities in the right hand side of (2.13) are well defined. This comes from an intermediate result which states that, under Assumptions 2.1 and 2.2, $\left|H_q(k, \frac{\partial u}{\partial k}(k, w), w)\right| = O(|k - \kappa^*(w)|)$ for k in a neighborhood of $\kappa^*(w)$. The latter information implies that, with the optimal investment strategy, a firm starting with a capital $k_0 \neq \kappa^*(w)$ never reaches $\kappa^*(w)$, even though its capital does tend to $\kappa^*(w)$ as $t \to \infty$.

REMARK 2.5. *Note that (2.13) implies that $\frac{1}{\nu\hat{c}} \leqslant \int_{\mathbb{R}_+} m(k, w) dk \leqslant \frac{\hat{c}}{\nu}$ (the two bounds do not depend on w). Moreover, the support of $m(\cdot, w)$ is contained in the interval*

$$[\min(\min\{k \in \mathit{support}(\hat{\eta})\}, \kappa^*(w)), \max(\max\{k \in \mathit{support}(\hat{\eta})\}, \kappa^*(w))].$$

Hence, from the continuity of $w \mapsto \kappa^*(w)$, for any compact set $K \subset \mathbb{R}_+^d$, there exists a compact interval of $\mathbb{R}_+$ containing the supports of $m(\cdot, w)$ for all $w \in K$.

2.3. Equilibria. The supply of inputs is assumed to be of the form $S(w)$, where $w \in \mathbb{R}_+^d$ is the collection of prices.

At the equilibrium, we require that the clearing condition on the markets of inputs holds, i.e.

$$S(w) = \int_{\mathbb{R}_+} \ell^* (k, w)\, m(k, w) dk. \tag{2.15}$$

where $\ell_i^* (k, w) = -\frac{\partial f}{\partial w_i}(k, w)$, and $m(\cdot, w)$ is the solution of (2.11)-(2.12).
We aim at proving the existence of equilibria by using Brouwer degree theory. This requires additional assumptions:

Assumption 2.4 (Assumptions on the supply). *The function $S : \mathbb{R}_+^d \to \mathbb{R}_+^d$ is of the form $S(w) = D_w\Phi(w)$, where*

1. *$\Phi : \mathbb{R}_+^d \to \mathbb{R}$ is C^1 regular and strictly convex*
2. *Φ is bounded from below (for example by 0)*
3. *Φ is coercive, i.e. $\lim_{\|w\|\to\infty} \Phi(w) = +\infty$.*

Remark 2.6. *The Legendre-Fenchel transform of Φ, $\Phi^*(S) = \sup_{w\in\mathbb{R}_+^d} S \cdot w - \Phi(w)$ is convex and semi-continuous on $\mathbb{R}_+^d$ with values in $(-\infty, +\infty]$. It can be interpreted as a collective cost or disutility associated to the supply of inputs. Concerning raw materials, it may be linked to their scarsity or to the environmental/social damages caused by their production. For manpower, the disutility captures negative effects of labour on the welfare of the workers.*

Examples.

1. If, for any $i = 1, \dots, d$, the i-th component of the supply function is a non negative, continuous and increasing function of w_i, i.e. $S_i(w) = S_i(w_i)$, then Assumption 2.4 is satisfied with $\Phi(w) = \sum_{i=1}^d \int_0^{w_i} S_i(t)dt$.
2. Given two positive numbers σ and w_0, if

$$S_i(w) = \frac{\exp(w_i/\sigma)}{\sum_{j=0}^d \exp(w_j/\sigma)}$$

for all $i = 1, \dots, d$,
then Assumption 2.4 is satisfied with $\Phi(w) = \sigma \log \left(\sum_{j=0}^d \exp(w_j/\sigma) \right)$. In the limit $\sigma \to 0$, the price w_0 can be seen as a reserve price under which the production factors cannot be acquired.

Set

$$g(k, w) = f(k, w) + \delta k = \sup_{\ell\in\mathbb{R}_+^d} F(k, \ell) - w \cdot \ell, \tag{2.16}$$

which can be seen as the Legendre-Fenchel transform of $\ell \mapsto -F(k, \ell)$ evaluated at $-w$. From Assumption 2.2, $g(k, w)$ is finite, nonnegative and achieved by the unique maximizer $\ell^*(k, w) \in \mathbb{R}_+^d$, and ℓ^* is C^1 on $(0, +\infty) \times (0, +\infty)^d$.

A further technical assumption involving both F and the fixed measure $\hat{\eta}$ arising in Assumption 2.3 will be needed:

Assumption 2.5. *Let $\mathbb{1} \in \mathbb{R}^d$ be defined by $\mathbb{1} = (1, \dots, 1)$. We assume that there exists $\epsilon \in (0, 1)$ such that for all $\lambda \in [0, 1]$, if*

$$\begin{aligned} &\Phi(w) + \int_0^{+\infty} g(k, w)\Big((1-\lambda)d\hat{\eta}(k) + \lambda dm(k, w)\Big) \\ \leqslant &\Phi(\mathbb{1}) + \int_0^{+\infty} g(k, \mathbb{1})\Big((1-\lambda)d\hat{\eta}(k) + \lambda dm(k, w)\Big) \end{aligned} \tag{2.17}$$

then

$$w \in \left(\epsilon, \frac{1}{\epsilon}\right)^d. \tag{2.18}$$

REMARK 2.7. *The proof that Assumption 2.5 holds for classical examples of production functions will be given in Section 5.*

Section 5 will be devoted to the proof of the following existence result:

THEOREM 2.3 (Existence of equilibria). *Under Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5, there exists an equilibrium, i.e.* $w \in (0, +\infty)^d$ *such that the market clearing condition* (2.15) *holds with* $m(\cdot, w)$ *and* $u(\cdot, w)$ *uniquely defined respectively by Proposition 2.2 and Theorem 2.1.*

2.4. Classical examples of utility and production functions.

2.4.1. Examples of utility functions. The constant relative risk aversion (CRRA) utility is a common example of a utility that satisfies Assumption 2.1:

$$U(c) = \ln(c) \quad \text{or} \quad U(c) = \frac{1}{b}c^b \quad \text{with } b \in (0,1)$$

2.4.2. Examples of production functions.

1. A classical example is the Cobb-Douglas function:

$$F(k, \ell) = Ak^\alpha \ell^\beta,$$

where $\beta \in (0,1)^d$, $\sum_{i=1}^d \beta_i < 1$, $\ell^\beta = \prod_{i=1}^d \ell_i^{\beta_i}$, and $0 < \alpha < 1 - \sum_{i=1}^d \beta_i$. Let us set $|\beta| = \sum_{i=1}^d \beta_i$. In this example, the parameters β and α respectively stand for the elasticities of the output with respect to the different inputs and to the capital, and $A > 0$ is a global factor of productivity. The net output is given by

$$f(k, w) = (1 - |\beta|) \left(Ak^\alpha \prod_{i=1}^d \left(\frac{\beta_i}{w_i}\right)^{\beta_i} \right)^{\frac{1}{1-|\beta|}} - \delta k. \tag{2.19}$$

It can be checked that the first order partial derivatives of f with respect to k and w_i are

$$\frac{\partial f}{\partial k}(k, w) = \alpha \left(A \prod_{j=1}^d \left(\frac{\beta_j}{w_j}\right)^{\beta_j} \right)^{\frac{1}{1-|\beta|}} k^{-\frac{1-\alpha-|\beta|}{1-|\beta|}} - \delta, \tag{2.20}$$

and

$$\frac{\partial f}{\partial w_i}(k, w) = -\left(Ak^\alpha \prod_{j=1}^d \left(\frac{\beta_j}{w_j}\right)^{\beta_j} \right)^{\frac{1}{1-|\beta|}} \frac{\beta_i}{w_i} \leqslant 0. \tag{2.21}$$

It is easy to see that Assumption 2.2 is satisfied. In particular, $\lim_{k\to+\infty} \frac{\partial f}{\partial k}(k, w) = -\delta$. The capital $\kappa^*(w)$ in (2.10) is given by

$$\kappa^*(w) = \left(\frac{\alpha}{\alpha+\rho}\right)^{\frac{1-|\beta|}{1-\alpha-|\beta|}} \left(A \prod_{j=1}^d \left(\frac{\beta_j}{w_j}\right)^{\beta_j} \right)^{\frac{1}{1-\alpha-|\beta|}}. \tag{2.22}$$

2. We now consider a production function with a constant elasticity of substitution:

$$F(k,\ell) = \left(k^\alpha + \sum_{i=1}^d \ell_i^{\beta_i}\right)^\gamma,$$

where $\alpha \in (0,1)$, $\beta \in (0,1)^d$ and $\gamma \in (0,1)$. For any $(k,w) \in \mathbb{R}_+ \times (0,+\infty)^d$, it can be checked that there exists a unique parameter $\lambda(k,w) > 0$ such that

$$\lambda \left(k^\alpha + \sum_{j=1}^d \left(\frac{\lambda\beta_j}{w_j}\right)^{\frac{\beta_j}{1-\beta_j}}\right)^{1-\gamma} = \gamma. \tag{2.23}$$

The net output is then

$$f(k,w) = \left(k^\alpha + \sum_{j=1}^d \left(\frac{\lambda(k,w)\beta_j}{w_j}\right)^{\frac{\beta_j}{1-\beta_j}}\right)^\gamma - \sum_{j=1}^d w_i \left(\frac{\lambda(k,w)\beta_j}{w_j}\right)^{\frac{1}{1-\beta_j}} - \delta k.$$

It can be checked that the first order partial derivatives of f with respect to k and w_i are

$$\frac{\partial f}{\partial k}(k,w) = \alpha\lambda(k,w)k^{\alpha-1} - \delta, \tag{2.24}$$

and

$$\frac{\partial f}{\partial w_i}(k,w) = -\left(\frac{\lambda(k,w)\beta_i}{w_i}\right)^{\frac{1}{1-\beta_i}} < 0. \tag{2.25}$$

Assumption 2.2 is satisfied. In particular, $\lim_{k\to+\infty} \frac{\partial f}{\partial k}(k,w) = -\delta$. The capital $\kappa^*(w)$ in (2.10) is the unique solution of

$$\alpha\lambda(\kappa^*(w),w)(\kappa^*(w))^{\alpha-1} = \delta + \rho. \tag{2.26}$$

3. The optimal control problem of a single firm. In this section, we assume that w, the prices of the production factors, is a fixed vector in $(0,+\infty)^d$. Thus, in order to alleviate the notation, we everywhere omit the dependency upon w; for example we write $H(k,q)$ and $u(k)$ instead of $H(k,q,w)$ and $u(k,w)$. Similarly, we set $u'(k) = \frac{\partial u}{\partial k}(k,w)$ and $f'(k) = \frac{\partial f}{\partial k}(k,w)$.

The proof of Theorem 2.1 is simpler when $\delta = 0$ because f is positive on $(0,+\infty)$. We will first focus on the latter case, then we will address the other case, i.e. $\delta > 0$.

3.1. The particular case where $\delta = 0$.

3.1.1. Some properties of the Hamiltonian. LEMMA 3.1. *Under Assumption 2.1, for any $k > 0$, the function $q \mapsto H(k,q)$, defined on $(0,+\infty)$, is strictly convex and of class C^2.*

Proof. From Assumption 2.1, the function U' is one to one on $(0,+\infty)$. Let c^* denote the inverse function, which is decreasing and C^1 on $(0,+\infty)$; its derivative is $q \mapsto 1/U''(c^*(q))$. For any $q > 0$, $c^*(q) > 0$ is the unique consumption which achieves the supremum in (2.5), because $U'(c^*(q)) = q$. The derivative of $q \mapsto H(k,q)$ is

$$H_q(k,q) = -c^*(q) + f(k). \tag{3.1}$$

Hence, $q \mapsto H(k,q)$ is C^2 on $(0,+\infty)$ and $H_{qq}(k,q) = -1/U''(c^*(q)) > 0$. This implies the strict convexity of $q \mapsto H(k,q)$. ∎

Remark 3.1. *Note that the consumption achieving the supremum in (2.5) does not depend on k.*

Lemma 3.2. *We make Assumptions 2.1 and 2.2 and suppose furthermore that $\delta = 0$, hence $\lim_{k\to+\infty} f'(k) = 0$. Then, for any $k > 0$,*

$$\min_{q>0} H(k,q) = U(f(k)), \tag{3.2}$$

$$\arg\min_{q>0} H(k,q) = \left\{U'(f(k))\right\}. \tag{3.3}$$

Proof. For $k > 0$, $f(k) > 0$ by Remark 2.2. From (3.1), $H_q(k,q) = 0$ if and only if $c^*(q) = f(k)$, i.e. $q = U'(f(k))$. This proves that the infimum of the strictly convex function $q \mapsto H(k,q)$ is a minimum, which is achieved by $q = U'(f(k))$. The minimal value is $U(c^*(U'(f(k)))) = U(f(k))$. ∎

Remark 3.2. *From Assumption 2.1, we see that if $f(0) = 0$, then $\lim_{k\to 0} U'(f(k)) = +\infty$. On the contrary, from Remark 2.2, if $f(0) > 0$, then $U' \circ f$ remains bounded on bounded subsets of $[0,+\infty)$.*

Lemma 3.3. *Under Assumption 2.1,*

$$\lim_{q\to 0+} H(k,q) = \lim_{c\to+\infty} U(c) - cU'(c) = \lim_{c\to+\infty} U(c) \in (-\infty,+\infty], \tag{3.4}$$

$$\lim_{q\to 0+} H_q(k,q) = -\infty. \tag{3.5}$$

Proof. Since c^* is the inverse of U' on $(0,+\infty)$, Assumption 2.1 implies that $\lim_{q\to 0} c^*(q) = +\infty$. Therefore, from (3.1), $\lim_{q\to 0} H_q(k,q) = -\infty$.

We know that U is increasing: let us set $\ell_1 = \lim_{c\to+\infty} U(c) = \sup_{c\geqslant 0} U(c) \in (-\infty,+\infty]$. On the other hand, the function $c \mapsto U(c) - cU'(c)$ is increasing in $\mathbb{R}_+$, because its derivative is $c \mapsto -cU''(c)$; let us set $\ell_2 = \lim_{c\to+\infty} U(c) - cU'(c) \in (-\infty,+\infty]$.

Since $H(k,q) \sim U(c^*(q)) - c^*(q)U'(c^*(q))$ as $q \to 0$, we see that $\lim_{q\to 0} H(k,q) = \ell_2$.

We need to compare ℓ_1 and ℓ_2. It is obvious that $\ell_2 \leqslant \ell_1$. We wish to prove that $\ell_2 = \ell_1$. We argue by contradiction and assume that $\ell_2 < \ell_1$. We make out two cases:

1. $\ell_1 \in \mathbb{R}$ and $\ell_2 < \ell_1$: we see that $cU'(c)$ tends to $\ell_1 - \ell_2 > 0$ as c tends to $+\infty$. This implies that $U(c)$ blows up like a logarithm of c as c tends to $+\infty$, in contradiction with the fact that $\ell_1 < +\infty$. Therefore, if ℓ_1 is finite, then $\ell_1 = \ell_2$.
2. $\ell_1 = +\infty$ and $\ell_2 \in \mathbb{R}$. We see that $cU'(c) = U(c) - \ell_2 + o(1)$ where $\lim_{c\to\infty} o(1) = 0$. Using Gronwall lemma, we deduce that there exists a real number χ such that $U(c) = \chi c + \ell_2 + o(1)$. Since $U(c) \to +\infty$ as $c \to +\infty$, we see that $\chi > 0$. We deduce that $\lim_{c\to\infty} U'(c) = +\infty$ in contradiction with Assumption 2.1.

The proof is complete. ∎

Lemmas 3.1 and 3.2 above allow us to define the increasing and decreasing parts of the Hamiltonian:

Definition 3.4. *We make Assumptions 2.1 and 2.2 and suppose furthermore that $\delta = 0$.*

- *Define the sets*

$$\Theta^{\uparrow} = \left\{(k,q) \text{ such that } k>0 \text{ and } q \geqslant U'(f(k))\right\},$$
$$\Theta^{\downarrow} = \left\{(k,q) \text{ such that } k>0 \text{ and } q \leqslant U'(f(k))\right\}.$$

- *Let $H^{\uparrow}(\cdot,\cdot)$ be the restriction of $H(\cdot,\cdot)$ to $\Theta^{\uparrow}$. The function $q \mapsto H^{\uparrow}(k,q)$ is increasing in $[U'(f(k)),\infty)$.*
- *Let $H^{\downarrow}(\cdot,\cdot)$ be the restriction of $H(\cdot,\cdot)$ to $\Theta^{\downarrow}$. The function $q \mapsto H^{\downarrow}(k,q)$ is decreasing on $(0,U'(f(k))]$.*

The graphs of $H(k,\cdot)$, $H^{\uparrow}(k,\cdot)$ and $H^{\downarrow}(k,\cdot)$ are displayed on Figure 3.1.

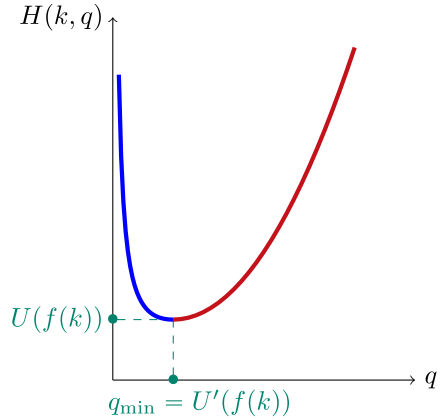

FIGURE 3.1. *The bold line (blue and red) is the graph of the function $H(k,\cdot)$. The blue line is the graph of $H^{\downarrow}(k,\cdot)$. The red line is the graph of $H^{\uparrow}(k,\cdot)$. In the present figure, $\lim_{q\to 0_+} H(k,q) = +\infty$, but it is also possible that $\lim_{q\to 0_+} H(k,q) \in \mathbb{R}$.*

LEMMA 3.5. *Under the same assumptions as in Lemma 3.2, $H^{\downarrow}(\cdot,\cdot)$ (respectively $H^{\uparrow}(\cdot,\cdot)$) is of class C^1 on $\Theta^{\downarrow}$ (respectively $\Theta^{\uparrow}$).*

Proof. We have already seen in the proof of Lemma 3.1 that $q \mapsto H(k,q)$ is of class C^2. Moreover, from Assumption 2.2, $k \mapsto f(k)q$ is of class C^1, so $k \mapsto H(k,q)$ is also of class C^1. Hence $(k,q) \mapsto H^{\downarrow}(k,q)$ is of class C^1 on $\Theta^{\downarrow}$, and so is $(k,q) \mapsto H^{\uparrow}(k,q)$ on $\Theta^{\uparrow}$. ∎

3.1.2. General orientation. Heuristically, if u is a classical solution of (2.7) such that $u'(k) > 0$ for $k > 0$ and u'' is locally bounded, then, taking the derivative of (2.7), we get that for $k > 0$,

$$\left(f'(k) - \rho\right) u'(k) = -H_q\left(k, u'(k)\right) u''(k).$$

We deduce that if the optimal investment is 0, i.e. $H_q\left(k, u'(k)\right) = 0$, then

$$f'(k) = \rho. \tag{3.6}$$

From Assumption 2.2, (3.6) has a unique solution which we name κ^* (note that κ^* depends on w, see (2.10) in Theorem 2.1).
Moreover, $H_q\left(\kappa^*, u'(\kappa^*)\right) = 0$ implies that $u'(\kappa^*) = U'(f(\kappa^*))$ and $H(\kappa^*, u'(\kappa^*)) = U(f(\kappa^*))$, see Figure 3.1. Hence, from (2.7), we deduce that $u(\kappa^*) = U(f(\kappa^*))/\rho$.
On the other hand, because of the state constraint, we expect that $H_q\left(k, u'(k)\right)$ is positive for small values of k. Hence, we expect that for a classical state constrained solution u of (2.7),

$$H\left(k,u'(k)\right) = \begin{cases} H^{\uparrow}\left(k,u'(k)\right), & \text{if } k < \kappa^*, \\ H^{\downarrow}\left(k,u'(k)\right), & \text{if } k > \kappa^*. \end{cases}$$

Therefore, we are going to look for u as the solution of two ordinary differential equations in $(0,\kappa^*)$ and $(\kappa^*,+\infty)$ which respectively involve the inverse functions of $q\mapsto H^{\uparrow}(k,q)$ and $q\mapsto H^{\downarrow}(k,q)$, with the boundary condition

$$u(\kappa^*)=U(f(\kappa^*))/\rho.$$

In order to carry out this program, we need to consider the inverse functions of $q\mapsto H^{\uparrow}(k,q)$ and $q\mapsto H^{\downarrow}(k,q)$:

DEFINITION 3.6. *We make Assumptions 2.1 and 2.2 and suppose furthermore that* $\delta=0$*.*

- *Define the sets*

$$\Omega^{\uparrow}=\{(k,v):\ k\in(0,\kappa^*]\ \textit{ and } \rho v\in(U(f(k)),+\infty)\}, \tag{3.7}$$

$$\Omega^{\downarrow}=\left\{(k,v):\ k\in[\kappa^*,+\infty)\ \textit{ and } \rho v\in\left(U(f(k)),\lim_{q\to 0^+}H(k,q)\right)\right\}. \tag{3.8}$$

- *Set*

$$\mathcal{F}^{\uparrow}(k,v)=\left(H^{\uparrow}(k,\cdot)\right)^{-1}(\rho v),\qquad \textit{for } (k,v)\in\Omega^{\uparrow}, \tag{3.9}$$

$$\mathcal{F}^{\downarrow}(k,v)=\left(H^{\downarrow}(k,\cdot)\right)^{-1}(\rho v),\qquad \textit{for } (k,v)\in\Omega^{\downarrow}. \tag{3.10}$$

Program. Our program will be as follows:

1. Prove that the following Cauchy problem has a unique solution $u^{\downarrow}:[\kappa^*,+\infty)\to\mathbb{R}$:

$$\frac{du^{\downarrow}}{dk}(k)=\mathcal{F}^{\downarrow}(k,u^{\downarrow}(k)),\qquad \text{for } k\geqslant\kappa^*, \tag{3.11}$$

$$(k,u^{\downarrow}(k))\in\Omega^{\downarrow},\qquad \text{for } k>\kappa^*, \tag{3.12}$$

$$u^{\downarrow}(\kappa^*)=\frac{1}{\rho}U(f(\kappa^*)). \tag{3.13}$$

2. Prove that the following Cauchy problem has a unique solution $u^{\uparrow}:(0,\kappa^*]\to\mathbb{R}$:

$$\frac{du^{\uparrow}}{dk}(k)=\mathcal{F}^{\uparrow}(k,u^{\uparrow}(k)),\qquad \text{for } k\leqslant\kappa^*, \tag{3.14}$$

$$(k,u^{\uparrow}(k))\in\Omega^{\uparrow},\qquad \text{for } 0<k<\kappa^*, \tag{3.15}$$

$$u^{\uparrow}(\kappa^*)=\frac{1}{\rho}U(f(\kappa^*)). \tag{3.16}$$

3. Prove that the function u which coincides with $u^{\uparrow}$ on $[0,\kappa^*]$ and $u^{\downarrow}$ on $[\kappa^*,+\infty)$ is the solution of (2.7)-(2.9).

Before starting this program, let us state a useful lemma:

LEMMA 3.7. *Under the same assumptions as in Lemma 3.2,* $\mathcal{F}^{\downarrow}(\cdot,\cdot)$ *(respectively* $\mathcal{F}^{\uparrow}(\cdot,\cdot)$*) is of class* C^1 *on* $\Omega^{\downarrow}$ *(respectively* $\Omega^{\uparrow}$*).*

Proof. We skip the proof for brevity and refer to [17], which contains an extended version of the present paper.
□

3.1.3. The Cauchy problem (3.11)-(3.13). Let us first consider the maximal solution ϕ_λ of the following Cauchy problem:

$$\phi'_\lambda(k) = \mathcal{F}^\downarrow(k, \phi_\lambda(k)), \qquad \text{for } k \geqslant \kappa^*, \tag{3.17}$$
$$(k, \phi_\lambda(k)) \in \Omega^\downarrow, \tag{3.18}$$
$$\phi_\lambda(\kappa^*) = \lambda, \tag{3.19}$$

for λ such that $(\kappa^*, \lambda) \in \Omega^\downarrow$, see (3.8). Cauchy-Lipschitz theorem may be applied because $\mathcal{F}^\downarrow$ is regular enough on $\Omega^\downarrow$. After having proved the existence and uniqueness of ϕ_λ, we will let λ tend to $U(f(\kappa^*))/\rho$ and obtain that the sequence ϕ_λ converges to a solution of (3.11)-(3.13). One reason for not applying directly the standard existence results to the Cauchy problem with initial condition $\lambda = U(f(\kappa^*))/\rho$ is that $\mathcal{F}^\downarrow(\cdot,\cdot)$ is not regular at the boundary of $\Omega^\downarrow$. In particular, $v \mapsto \mathcal{F}^\downarrow(\kappa^*, v)$ is not Lipschitz continuous in the neighborhood of $(\kappa^*, U(f(\kappa^*))/\rho)$. Moreover, the point $(\kappa^*, U(f(\kappa^*))/\rho)$ belongs to the boundary of $\Omega^\downarrow$; this forbids the direct use of Cauchy-Peano-Arzelà theorem for obtaining the existence of a solution.

Proposition 3.8. *We make Assumptions 2.1 and 2.2 and suppose furthermore that $\delta = 0$. For every λ such that $(\kappa^*, \lambda) \in \Omega^\downarrow$, there exists a unique global solution ϕ_λ of (3.17)-(3.19) in $[\kappa^*, +\infty)$. The function ϕ_λ is increasing and strictly concave.*

Proof. Setting $\Theta(k) = (k, \phi_\lambda(k))$, it is convenient to rewrite (3.17)-(3.19) in the equivalent form: find $k \mapsto \Theta(k) \in \Omega^\downarrow$ such that

$$\Theta'(k) = \left(1, \mathcal{F}^\downarrow(\Theta(k))\right), \quad k \geqslant \kappa^*, \tag{3.20}$$
$$\Theta(\kappa^*) = (\kappa^*, \lambda). \tag{3.21}$$

We may apply Cauchy-Lipschitz theorem; indeed, from Lemma, the map $\Theta \mapsto \left(1, \mathcal{F}^\downarrow(\Theta)\right)$ is C^1 on $\Omega^\downarrow$. Therefore, there exists a unique maximal solution Θ of (3.20)-(3.21) in $[\kappa^*, \bar{k})$. We observe that for $k \in [\kappa^*, \overline{k})$, $\phi'_\lambda(k) = \mathcal{F}^\downarrow(k, \phi_\lambda(k)) > 0$, so $\lim_{k\to\bar{k}^-} \phi_\lambda(k)$ exists. Moreover, by taking the derivative,

$$\phi''_\lambda(k) = \frac{\rho - f'(k)}{H_q\left(k, \phi'_\lambda(k)\right)} \phi'_\lambda(k) < 0.$$

Therefore ϕ_λ is strictly concave in $[\kappa^*, \bar{k})$.

If $\bar{k} < \infty$, then from Cauchy-Lipschitz theorem, $\rho \lim_{k\to\bar{k}^-} \phi_\lambda(k)$ must be equal either to $U(f(\bar{k}))$ or to $\lim_{q\to 0} H(k,q) = \lim_{c\to+\infty} U(c)$ (which does not depend on k). Let us show by contradiction that both cases are impossible.

1. Assume first that $\rho \lim_{k\to\bar{k}^-} \phi_\lambda(k) = \lim_{q\to 0} H(k,q) = \lim_{c\to+\infty} U(c)$; let us make out two subcases:
   (a) If $\lim_{c\to+\infty} U(c) = +\infty$, then $\lim_{k\to\bar{k}^-} \phi_\lambda(k) = +\infty$, which yields that $\lim_{k\to\bar{k}^-} \mathcal{F}^\downarrow(k, \phi_\lambda(k)) = 0$. From (3.20), we see that $\lim_{k\to\bar{k}^-} \phi'_\lambda(k) = 0$, in contradiction with $\lim_{k\to\bar{k}^-} \phi_\lambda(k)) = +\infty$.
   (b) If $\lim_{c\to+\infty} U(c) = \ell \in \mathbb{R}$, then it is possible to extend continuously ϕ_λ to $\bar{k}$ by setting $\phi_\lambda(\bar{k}) = \ell/\rho$. Since $H(k, 0) = \ell$ for all k, we see that

$$\mathcal{F}^\downarrow(k, \ell/\rho) = 0, \qquad \text{for all } k \geqslant \kappa^*. \tag{3.22}$$

   On the other hand, since $U'(c^*(q)) = q$, Assumption 2.1 implies that $\lim_{q\to 0} c^*(q) = +\infty$. This implies that

$$\frac{\partial \mathcal{F}^\downarrow}{\partial v}(k, \ell/\rho) = 0, \qquad \text{for all } k \geqslant \kappa^*. \tag{3.23}$$

But (3.22) and (3.23) prevent the state ℓ/ρ to be reached in finite time by a solution of (3.17)-(3.18); we have obtained the desired contradiction.

2. Assume that $\lim_{k\to\bar{k}^-}\phi_\lambda(k) = U(f(\bar{k}))/\rho$. It is then possible to extend continuously ϕ_λ to $\bar{k}$ by setting $\phi_\lambda(\bar{k}) = U(f(\bar{k}))/\rho$, and (3.17) holds in $[\kappa^*, \bar{k}]$. On the other hand,

$$\frac{d}{dk}\left(\frac{U(f(k))}{\rho}\right) - \mathcal{F}^\downarrow\left(k, \frac{U(f(k))}{\rho}\right) = U'(f(k))\frac{f'(k)-\rho}{\rho} < 0, \quad \text{for } k > \kappa^*, \tag{3.24}$$

from the definition of κ^* and Assumption 2.2. Thus, $k \mapsto U(f(k))/\rho$ is a subsolution of the ordinary differential equation satisfied by ϕ_λ, which yields that $U(f(k))/\rho > \phi_\lambda(k)$ for $k < \bar{k}$. This is impossible, since $(k, \phi_\lambda(k)) \in \Omega^\downarrow$ for $k < \bar{k}$.

We have proved that $\bar{k} = +\infty$. The unique maximal solution of (3.20)-(3.21) is a global solution. □

Letting λ tend to $U(f(\kappa^*))/\rho$, we shall prove the following result:

Proposition 3.9. *Under the same assumptions as in Proposition 3.10, the Cauchy problem (3.11)-(3.13) has a unique solution $u^\downarrow \in C^1([\kappa^*, +\infty)) \cap C^2(\kappa^*, +\infty)$. Moreover $u^\downarrow$ is strictly concave on $(\kappa^*, +\infty)$.*

Proof. Consider a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$, such for all $n \in \mathbb{N}$, $(\kappa^*, \lambda_n) \in \Omega^\downarrow$ and $\lim_{n\to\infty}\lambda_n = U(f(\kappa^*))/\rho$. A direct consequence of Cauchy-Lipschitz theorem is that $\phi_{\lambda_n}(k) > \phi_{\lambda_{n+1}}(k)$ for all $k \geqslant \kappa^*$. On the other hand, we know that $\phi_{\lambda_n}(k) \geqslant U(f(k))/\rho$. This implies that there exists a function $\phi : [\kappa^*, +\infty) \to \mathbb{R}$ such that ϕ_{λ_n} converge to ϕ pointwise as n tends to $+\infty$.

Since $(\phi_{\lambda_n})_{n\in\mathbb{N}}$ is a sequence of concave functions locally uniformly bounded, we see from [8, Theorem 3.3.3] that the convergence is uniform on every compact set, so the limit ϕ is continuous.

Since $\mathcal{F}^\downarrow(\cdot,\cdot)$ is continuous on the closure of $\Omega^\downarrow$, we may pass to the limit in the integral form of (3.17): for all $k \geqslant \kappa^*$,

$$\phi(k) = \frac{1}{\rho}U(f(\kappa^*)) + \int_{\kappa^*}^{k} \mathcal{F}^\downarrow(\kappa, \phi(\kappa))d\kappa.$$

This implies that $\phi \in C^1([\kappa^*, +\infty))$ and that ϕ satisfies (3.11) and (3.13). Hence ϕ is an increasing function.

On the other hand, (3.24) implies that $\phi(k) > U(f(k))/\rho$ for all $k > \kappa^*$. This shows that ϕ satisfies (3.12).

Arguing as in the proof of Proposition 3.8, we see that ϕ is C^2 on $(\kappa^*, +\infty)$ and strictly concave. We have proved the existence of a solution of (3.11)-(3.13).

Assume that there are two such solutions ϕ_1 and ϕ_2. If there exists $k_0 > \kappa^*$ such that $\phi_1(k_0) = \phi_2(k_0)$, then ϕ_1 and ϕ_2 coincide from Cauchy-Lipschitz theorem. Hence we may assume that $\phi_1(k) < \phi_2(k)$ for $k > \kappa^*$. Then, using the non increasing character of $\mathcal{F}^\downarrow(k,\cdot)$, we see that, for every $k > \kappa^*$,

$$0 > \phi_1(k) - \phi_2(k) = \int_{\kappa^*}^{k} \mathcal{F}^\downarrow(\kappa, \phi_1(\kappa)) - \mathcal{F}^\downarrow(\kappa, \phi_2(\kappa))d\kappa \geqslant 0.$$

We have found a contradiction and achieved the proof of uniqueness. □

3.1.4. The Cauchy problem (3.14)-(3.16)**.** Also in this case, $\mathcal{F}^\uparrow(k,\cdot)$ is not Lipschitz continuous in the neighborhood of $(\kappa^*, U(f(\kappa^*))/\rho)$ and $(\kappa^*, U(f(\kappa^*))/\rho)$

belongs to the boundary of $\Omega^{\uparrow}$. This prevents us from applying directly standard existence results to (3.14)-(3.16).
For this reason, we start by considering the Cauchy problem:

$$\psi'_{\epsilon,\lambda}(k) = \mathcal{F}^{\uparrow}(k, \psi_{\epsilon,\lambda}(k)), \quad 0 < k \leqslant \kappa^*, \tag{3.25}$$

$$(k, \psi_{\epsilon,\lambda}(k)) \in \Omega^{\uparrow}, \tag{3.26}$$

$$\psi_{\epsilon,\lambda}(\epsilon) = \lambda, \tag{3.27}$$

for $(\epsilon, \lambda) \in \Omega^{\uparrow}$, see (3.10) (thus $0 < \epsilon < \kappa^*$). As above, Cauchy-Lipschitz theorem may be applied to (3.25)-(3.27). After having obtained the existence and uniqueness of a maximal solution $\psi_{\epsilon,\lambda}$, we will prove that there exists λ such that $\psi_{\epsilon,\lambda}$ is a global solution, i.e. defined on $(0, \kappa^*]$, and that $\psi_{\epsilon,\lambda}(\kappa^*) = U(f(\kappa^*))/\rho$.

Lemma 3.10. *We make Assumptions 2.1 and 2.2 and suppose furthermore that $\delta = 0$. For every $(\epsilon, \lambda) \in \Omega^{\uparrow}$ with $0 < \epsilon < \kappa^*$, there exists a unique maximal solution of the Cauchy problem* (3.25)-(3.27) *of the form $\left((0, \overline{k}(\epsilon, \lambda)), \psi_{\epsilon,\lambda}\right)$ where $\epsilon < \overline{k}(\epsilon, \lambda) \leqslant \kappa^*$. The function $\psi_{\epsilon,\lambda}$ is strictly concave and increasing in $(0, \overline{k}(\epsilon, \lambda))$.*

Proof. Existence and uniqueness of a maximal solution follow from the Cauchy-Lipschitz theorem. The strict monotonicity and concavity of $\psi_{\epsilon,\lambda}$ are obtained as in Proposition 3.8. Assume by contradiction that the interval in the definition of the maximal solution is not of the form $(0, \overline{k}(\epsilon, \lambda))$. This implies that there exists $\underline{k} \in (0, \epsilon)$ such that either $\lim_{k \to \underline{k}} \psi_{\epsilon,\lambda}(k) = -\infty$ or $\psi_{\epsilon,\lambda}(\underline{k}) = U(f(\underline{k}))/\rho$. Let us rule out both situations:

- If $\lim_{k \to \underline{k}} \psi_{\epsilon,\lambda}(k) = -\infty$, then $\lim_{k \to \underline{k}} \psi'_{\epsilon,\lambda}(k) = +\infty$. This implies that $\lim_{k \to \underline{k}} \psi_{\epsilon,\lambda}(k) = U(f(\underline{k}))/\rho$, and we have obtained the desired contradiction.
- If $\psi_{\epsilon,\lambda}(\underline{k}) = U(f(\underline{k}))/\rho$, then proceeding as in the end of the proof of Proposition 3.8, this implies that $\psi_{\epsilon,\lambda}(k) \leqslant U(f(k))/\rho$ for all $k \in [\underline{k}, \epsilon]$, in contradiction with $\psi_{\epsilon,\lambda}(\epsilon) = \lambda > U(f(\epsilon))/\rho$.

Therefore the maximal solution is defined in an interval of the form $(0, \overline{k}(\epsilon, \lambda))$. □

Remark 3.3. *Note that if $f(0) = 0$, then $\psi'_{\epsilon,\lambda}(k)$ blows up when $k \to 0^+$: indeed, from* (3.1), $0 < H_q(k, \psi'_{\epsilon,\lambda}(k)) = f(k) - c^*\left(\psi'_{\epsilon,\lambda}(k)\right)$, *hence* $c^*\left(\psi'_{\epsilon,\lambda}(k)\right) < f(k)$. *Therefore,* $U'\left(c^*\left(\psi'_{\epsilon,\lambda}(k)\right)\right) > U'(f(k))$. *Thus, from Assumption 2.1,* $\psi'_{\epsilon,\lambda}(k) = U'\left(c^*\left(\psi'_{\epsilon,\lambda}(k)\right)\right) > U'(f(k))$ *tends to $+\infty$ as $k \to 0$.*

Lemma 3.11. *Under the same assumptions as in Lemma 3.10, for every $\epsilon \in (0, \kappa^*)$, the set*

$$\Lambda_\epsilon = \left\{\lambda > U(f(\epsilon))/\rho \quad \textit{such that} \quad \overline{k}(\epsilon, \lambda) = \kappa^*\right\} \tag{3.28}$$

is not empty.

Proof. Take $\lambda > U(f(\kappa^*))/\rho$. Assume by contradiction that $\overline{k}(\epsilon, \lambda) < \kappa^*$, where $\left((0, \overline{k}(\epsilon, \lambda)), \psi_{\epsilon,\lambda}\right)$ is the maximal solution of the Cauchy problem (3.25)-(3.27), (note that $\epsilon < \overline{k}(\epsilon, \lambda)$).
Observe first that $\psi_{\epsilon,\lambda}$ cannot blow up as $k \to \overline{k}(\epsilon, \lambda)$. Indeed $v \mapsto \mathcal{F}^{\uparrow}(k, \rho v)$ is Lipschitz continuous on $[\max_{k \in [\epsilon, \kappa^*]} U(f(k)) + 1, +\infty)$ with a Lipschitz constant that does not depend on $k \in [\epsilon, \kappa^*]$. This property prevents $\psi_{\epsilon,\lambda}$ from blowing up in finite time.
Therefore, the function $\psi_{\epsilon,\lambda}$ can be extended to $\overline{k}(\epsilon, \lambda)$ by continuity, and

$$\psi_{\epsilon,\lambda}(\overline{k}(\epsilon, \lambda)) = U(f(\overline{k}(\epsilon, \lambda)))/\rho, \tag{3.29}$$

otherwise it would not be the maximal solution. On the other hand, we know that f is increasing in $(0, \kappa^*]$, hence $U(f(\kappa^*)) > U(f(k))$ for all $k < \kappa^*$. In particular, $U(f(\kappa^*)) > U(f(\overline{k}(\epsilon, \lambda)))$. From the monotonicity of $\psi_{\epsilon,\lambda}$, we obtain that

$$\psi_{\epsilon,\lambda}(\overline{k}(\epsilon, \lambda)) \geqslant \psi_{\epsilon,\lambda}(\epsilon) = \lambda > U(f(\kappa^*))/\rho > U(f(\overline{k}(\epsilon, \lambda)))/\rho,$$

which contradicts (3.29).
We have proved that if $\lambda > U(f(\kappa^*))/\rho$, then the maximal solution is defined on $(0, \kappa^*]$. Therefore, Λ_ϵ is not empty. ◻

Proposition 3.12. *For all $\epsilon < \kappa^*$, there exists λ such that $(\epsilon, \lambda) \in \Omega^\uparrow$ and a global solution $\psi_{\epsilon,\lambda}$ (i.e. defined on $(0, \kappa^*]$) of the Cauchy problem (3.25)-(3.27) such that $\psi_{\epsilon,\lambda}(\kappa^*) = U(f(\kappa^*))/\rho$.*

Proof. Consider a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ_ϵ (see (3.28)) such that $\lim_{n\to\infty} \lambda_n = \underline{\lambda}_\epsilon = \inf_{\lambda\in\Lambda_\epsilon} \lambda$. It is clear that $(\psi_{\epsilon,\lambda_n})_{n\in\mathbb{N}}$ is a decreasing sequence of functions defined on $(0, \kappa^*]$. Moreover, since $(k, \psi_{\epsilon,\lambda_n}(k)) \in \Omega^\uparrow$ for $k \in (0, \kappa^*)$, $\psi_{\epsilon,\lambda_n}$ is bounded from below by the function $U \circ f/\rho$. Hence, there exists a function ψ_ϵ defined on $(0, \kappa^*]$ such that $\lim_{n\to+\infty} \psi_{\epsilon,\lambda_n}(k) = \psi_\epsilon(k)$ for all $k \in (0, \kappa^*]$.
Since $(\psi_{\epsilon,\lambda_n})_{n\in\mathbb{N}}$ is a sequence of concave functions locally uniformly bounded, [8, Theorem 3.3.3] ensures that the convergence is uniform on every compact set, thus ψ_ϵ is continuous on $(0, \kappa^*]$. Extending $\mathcal{F}^\uparrow(\cdot, \cdot)$ by continuity on the set $\{(k, U(f(k))/\rho) : k \in (0, \kappa^*]\}$, we may pass to the limit in the integral form of the differential equation satisfied by $\psi_{\epsilon,\lambda_n}$ and get

$$\psi_\epsilon(k) = \underline{\lambda}_\epsilon + \int_\epsilon^k \mathcal{F}^\uparrow(\kappa, \psi_\epsilon(\kappa))d\kappa.$$

Hence ψ_ϵ is a solution of (3.25) on $(0, \kappa^*)$, which implies that ψ_ϵ is C^1 and increasing in $(0, \kappa^*)$.
We are left with proving that $\psi_\epsilon(\kappa^*) = U(f(\kappa^*))/\rho$. It is already known that $\psi_\epsilon(\kappa^*) \geqslant U(f(\kappa^*))/\rho$. Assume by contradiction that $\psi_\epsilon(\kappa^*) > U(f(\kappa^*))/\rho$. Then, set

$$b = \frac{\psi_\epsilon(\kappa^*) + U(f(\kappa^*))/\rho}{2},$$

and consider the Cauchy problem on $(0, \kappa^*]$:

$$\begin{aligned} \xi'(k) &= \mathcal{F}^\uparrow(k, \xi(k)), \\ (k, \xi(k)) &\in \Omega^\uparrow, \\ \xi(\kappa^*) &= b. \end{aligned}$$

It can be proved by contradiction (with the same kind of argument as in the end of the proof of Proposition 3.8) that the maximal solution of this problem is in fact global, therefore defined on $(0, \kappa^*]$. Cauchy-Lipschitz theorem implies that $\xi(k) < \psi_\epsilon(k)$ for all $k \in (0, \kappa^*]$. Therefore, $\xi(\epsilon) \in \Lambda_\epsilon$ and $\xi(\epsilon) < \psi_\epsilon(\epsilon) = \underline{\lambda}_\epsilon$, which contradicts the definition of $\underline{\lambda}_\epsilon$.
Therefore, $\psi_\epsilon(\kappa^*) = U(f(\kappa^*))/\rho$. The same arguments as in the proof of Proposition 3.8 yield that $\psi_\epsilon(k) > U(f(k))/\rho$ for all $k \in (0, \kappa^*)$. Hence $\psi_\epsilon = \psi_{\epsilon,\underline{\lambda}_\epsilon}$. This achieves the proof. ◻

Proposition 3.13. *Under the same assumptions as in Lemma 3.10, the Cauchy problem (3.14)-(3.16) has a unique solution $u^\uparrow \in C^1((0, \kappa^*]) \cap C^2(0, \kappa^*)$. Moreover $u^\uparrow$ is strictly concave on $(0, \kappa^*)$.*

Proof. Existence is a consequence of Proposition 3.12. Uniqueness is proved exactly with the same arguments as in the proof of Proposition 3.9. □

REMARK 3.4. *From Remark 3.3, it is possible that* $\lim_{k\to 0} \frac{du^\uparrow}{dk}(k) = +\infty$ *and that* $\lim_{k\to 0} u^\uparrow(k) = -\infty$.

3.1.5. End of the proof of Theorem 2.1 in the particular case where $\delta = 0$.

Existence. With $u^\uparrow$ and $u^\downarrow$ as in Propositions 3.13 and 3.9, define

$$u(k) = \begin{cases} u^\uparrow(k), & \text{if } k \in (0, \kappa^*], \\ u^\downarrow(k), & \text{if } k \in [\kappa^*, +\infty). \end{cases} \tag{3.30}$$

The properties of $u^\uparrow$ and $u^\downarrow$ ensure that u is of class C^1, increasing and strictly concave in $(0, +\infty)$, and C^2 in $(0, \kappa^*) \cup (\kappa^*, +\infty)$. In particular, $u^\uparrow(\kappa^*) = u^\downarrow(\kappa^*) = \frac{1}{\rho} U(f(\kappa^*))$ and $\frac{du^\uparrow}{dk}(\kappa^*) = \frac{du^\downarrow}{dk}(\kappa^*) = U'(f(\kappa^*))$. Moreover,

$$H_q\left(k, u'(k)\right) \begin{cases} > 0, & \text{if } k \in (0, \kappa^*), \\ < 0, & \text{if } k \in (\kappa^*, +\infty), \\ = 0 & \text{if } k = \kappa^*. \end{cases}$$

Hence, u satisfies (2.7)-(2.9).

Uniqueness and characterization by (2.3). Let us now prove that if $u \in C^1(0, +\infty) \cap C^2((0, \kappa^*) \cup (\kappa^*, +\infty))$ satisfies (2.7)-(2.9), then it is the value function of problem (2.2). This will yield the uniqueness of a classical solution of (2.7)-(2.9) as well as the characterization of the value function of (2.2).

Let us set $\chi(\cdot) = c^*(u'(\cdot)) = f(\cdot) - H_q(\cdot, u'(\cdot))$. Assumptions 2.1, 2.2, and Lemma 4.1 below imply that $k \mapsto H_q(\cdot, u'(\cdot))$ is locally Lipschitz continuous on $(0, +\infty)$. This property and (2.8)-(2.9) imply that for any $k_0 \in (0, +\infty)$, there is a unique solution k of the Cauchy problem

$$\begin{aligned} \frac{dk}{dt}(t) &= f\left(k(t)\right) - \chi\left(k(t)\right), \quad t > 0 \\ k(0) &= k_0, \end{aligned}$$

It is an admissible trajectory for problem (2.2). Therefore u is not greater than the value function of the optimal control problem (2.3).

On the other hand, consider $c \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+)$, $\ell \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^d_+)$, $k \in W^{1,1}_{\text{loc}}(\mathbb{R}_+)$, such that

$$\begin{aligned} \frac{dk}{dt}(t) = F(k(t), \ell(t)) - w \cdot \ell(t) - \delta k(t) - c(t), &\quad \text{for a.a. } t > 0, \\ k(0) = k_0&, \\ k(t) \geqslant 0, &\quad \text{for a.a. } t > 0. \end{aligned}$$

Observe that for almost every $t \geqslant 0$,

$$\begin{aligned} &\sup_{\bar{c} \geqslant 0, \bar{l} \geqslant 0} \left\{ U(\bar{c}) + u'(k(t)) \left(F(k(t), \bar{l}) - w\bar{l} - \delta k(t) - \bar{c} \right) \right\} \\ \geqslant & U(c(t)) + u'(k(t)) \left(F(k(t), \ell(t)) - w\ell(t) - \delta k(t) - c(t) \right) \\ = & U(c(t)) + u'(k(t)) \frac{dk}{dt}(t). \end{aligned}$$

The left hand side coincides with $H(k(t), u'(k(t))) = \rho u(k(t))$. Hence, $U(c(t)) \leqslant -u'(k(t))\frac{dk}{dt}(t) + \rho u(k(t))$. This implies that $\int_0^\infty U(c(t))e^{-\rho t}dt \leqslant u(k_0)$. Hence, u is not smaller than the value function of problem (2.3).
We have proved that if $u \in C^1(0, +\infty)$ satisfies (2.7)-(2.9), then it is the value function of problem (2.2).

3.2. The case where $\delta > 0$.

LEMMA 3.14. *We make Assumption 2.2 and suppose furthermore that $\delta > 0$. Then there exits a unique $k_0 \in (0, +\infty)$ such that*

$$f(k_0) = 0. \tag{3.31}$$

The function f takes positive values on $(0, k_0)$ and negative values on $(k_0, +\infty)$. Moreover, $f'(k_0) < 0$ and $\kappa^* < k_0$, where κ^* is the unique positive number such that $f'(\kappa^*) = \rho$, see (2.10).

Proof. Since the proof is elementary, we skip it for brevity. ◻

Proof. [Proof of Theorem 2.1 when $\delta > 0$] Lemma 3.14 implies that in the interval $(0, k_0)$ which contains κ^* and where f is positive, it is possible to repeat the construction done in paragraphs 3.1.3 and 3.1.4. New arguments will be needed to construct the solution in $[k_0, +\infty)$.

Step 1. In $(0, k_0)$, it is possible to repeat the construction made in paragraphs 3.1.3 and 3.1.4: there exists a unique classical solution $u_1 \in C^1(0, k_0)$ of the following problem:

$$-\rho u_1(k) + H\left(k, u_1'(k)\right) = 0, \quad \text{for } 0 < k < k_0, \tag{3.32}$$

$$H_q\left(k, u_1'(k)\right) > 0, \quad \text{for } 0 < k < \kappa^*, \tag{3.33}$$

$$H_q\left(k, u_1'(k)\right) < 0, \quad \text{for } \kappa^* < k < k_0. \tag{3.34}$$

The function u_1 is strictly concave and increasing in $(0, k_0)$.
Since f is continuous and concave and $\lim_{k\to 0} f'(k) = +\infty$, $f'(k_0) < 0$, there exists $\bar{k} \in (\kappa^*, k_0)$ such that $f(\bar{k}) = \max_{k\in[0,k_0]} f(k)$. Since $u_1(\cdot)$ is increasing, $\lim_{k\to k_0} u_1(k_0) \geqslant u_1(\bar{k})$. On the other hand, $\rho u_1(\bar{k}) > U(f(\bar{k}))$ (see paragraph 3.1.3). Since U is increasing, $U(f((\bar{k})) > \lim_{k\to k_0} U(f(k)) = \lim_{c\to 0} U(c)$ (which may be $-\infty$). Therefore,

$$\rho \lim_{k\to k_0} u_1(k_0) > \lim_{c\to 0} U(c).$$

With the same kind of arguments as in the proof of Proposition 3.8, it can also be proved that $\rho u_1(k_0) < \lim_{c\to+\infty} U(c)$. This implies that $u_1(\cdot)$ can be extended by continuity to $(0, k_0]$ and that

$$\lim_{c\to 0} U(c) < \rho u_1(k_0) < \lim_{c\to\infty} U(c). \tag{3.35}$$

The function $u_1'(\cdot)$ can then be extended by continuity to $k = k_0$ and (3.32) holds up to $k = k_0$.

Step 2. We are left with constructing the solution in $(k_0, +\infty)$.
Observe first that, for any $k \geqslant k_0$, $q \mapsto H(k, q)$ is decreasing from (3.1), and that

1. $\lim_{q\to 0} H(k, q) = \lim_{c\to+\infty} U(c)$
2. Since $\lim_{q\to+\infty} c^*(q) = 0$ and $U(c) - cq + f(k)q \leqslant U(c)$, we deduce that

$$\lim_{q\to+\infty} H(k, q) \leqslant \lim_{c\to 0} U(c).$$

Hence, for any $k \geqslant k_0$, $q \mapsto H(k,q)$ maps $(0,+\infty)$ onto the interval $(\lim_{c\to 0} U(c), \lim_{c\to+\infty} U(c))$ and has a right inverse $z \mapsto \mathcal{F}(k,z)$:
for any $z \in (\lim_{c\to 0} U(c), \lim_{c\to+\infty} U(c))$, there is a unique $\mathcal{F}(k,z) > 0$ such that $H(k, \mathcal{F}(k,z)) = z$.

Let $\varepsilon > 0$ be small enough so that $\rho(u_1(k_0) - \varepsilon) > \lim_{c\to 0} U(c)$, see (3.35). Set

$$\Omega = \left\{ (k,v) \ : \ k_0 \leqslant k \text{ and } \rho(u_1(k_0) - \varepsilon) < \rho v < \lim_{c\to+\infty} U(c) \right\}. \tag{3.36}$$

Note that $(k_0, u_1(k_0)) \in \Omega$. It is possible to prove that $\mathcal{F}(\cdot,\cdot)$ is of class C^1 on Ω. Furthermore, it can be seen that $v \mapsto \mathcal{F}(k,v)$ is Lipschitz continuous on $[u_1(k_0) - \varepsilon, \lim_{c\to\infty} U(c)/\rho]$ with a Lipschitz constant which does not depend on $k \in [k_0, +\infty)$.

Consider the Cauchy problem

$$u_2'(k) = \mathcal{F}(k, u_2(k)), \qquad \text{for } k \geqslant k_0, \tag{3.37}$$

$$(k, u_2(k)) \in \Omega, \tag{3.38}$$

$$u_2(k_0) = u_1(k_0). \tag{3.39}$$

From Cauchy-Lipchitz theorem, there is a unique maximal solution of (3.37)-(3.39). The same arguments as in the proof of Proposition 3.8 yield that the solution is indeed global, i.e. defined on $[k_0, +\infty)$, increasing and strictly concave.

Step 3. Set

$$u(k) = \begin{cases} u_1(k), & \text{if } k \in (0, k_0], \\ u_2(k), & \text{if } k \in [k_0, +\infty). \end{cases}$$

From what precedes, $u \in C^1(0,+\infty)$, and $\rho u(k) = H(k, u'(k))$ for any $k \in (0,+\infty)$. Note that u is also C^2 in $(0,\kappa^*) \cup (\kappa^*, +\infty)$. Hence, u is a classical solution of (2.7)-(2.9). The remaining part of the proof (uniqueness and verification result) is exactly as in paragraph 3.1.5. ☐

4. The distribution of capital. We still assume that w, the prices of the production factors, is a fixed vector in $(0,+\infty)^d$; we keep omitting w everywhere. The optimal investment policy of a firm with capital k is $H_q(k, u'(k))$, where u is the solution of (2.7)-(2.9). We are interested in finding a weak solution m of the following problem:

$$\frac{d}{dk}\left(m H_q \left(\cdot, \frac{du}{dk}(\cdot) \right) \right) = \eta(\cdot, u(\cdot)) - \nu m(\cdot), \tag{4.1}$$

$$\nu \int_{\mathbb{R}_+} m(k) dk = \int_{\mathbb{R}_+} \eta(k, u(k)) dk. \tag{4.2}$$

From (2.8)-(2.9), we see that if (4.1) holds, then the optimal investment strategy has the effect of pushing m toward κ^*. It is therefore important to understand whether m has a singularity at $k = \kappa^*$. For that, the following lemma gives information on the behavior of u near κ^*:

Lemma 4.1. *Under Assumptions 2.1 and 2.2, there exist $\epsilon > 0$ and $M > 0$ such that*

$$0 \leqslant H_q(\kappa, u'(k)) \leqslant M(\kappa^* - k), \qquad \textit{if} \quad k \in [\kappa^* - \epsilon, \kappa^*], \tag{4.3}$$

$$M(\kappa^* - k) \leqslant H_q(k, u'(k)) \leqslant 0, \qquad \textit{if} \quad k \in [\kappa^*, \kappa^* + \epsilon]. \tag{4.4}$$

Proof. We focus on the proof of (4.3), since the proof of (4.4) is completely similar.
Since $u \in C^1(0,+\infty)$, and u is C^2 in $(0,\kappa^*) \cup (\kappa^*,+\infty)$, it is possible to take the derivative of (2.7) at $\kappa \neq \kappa^*$:

$$\rho u'(\kappa) - H_k(\kappa, u'(\kappa)) = H_q(\kappa, u'(\kappa))u''(\kappa). \tag{4.5}$$

Let us set

$$\chi(\kappa) = c^*(u'(\kappa)). \tag{4.6}$$

Note that $\chi(\kappa^*) = f(\kappa^*)$. The function χ is positive, continuous and increasing in $(0,+\infty)$, and C^1 on $(0,\kappa^*) \cup (\kappa^*,+\infty)$. Recall that

$$H_k(\kappa, u'(\kappa)) = f'(\kappa)u'(\kappa), \quad u'(\kappa) = U'(\chi(\kappa)), \quad \text{and} \quad H_q(\kappa, u'(\kappa)) = f(\kappa) - \chi(\kappa).$$

Then (4.5) can be written as follows:

$$U'(\chi(\kappa))(\rho - f'(\kappa)) = (f(\kappa) - \chi(\kappa))U''(\chi(\kappa))\chi'(\kappa). \tag{4.7}$$

The inequality on the left hand side of (4.3) is already known since $f(k)-\chi(k) > 0$ for $k < \kappa^*$. We are left with proving the other inequality for k sufficiently close to κ^*. We first claim that there exist $\epsilon > 0$ and $C_2 > 0$ such that for every $k \in [\kappa^* - \epsilon, \kappa^*]$,

$$\chi(\kappa^*) - \chi(k) = f(\kappa^*) - \chi(k) \leqslant C_2(\kappa^* - k). \tag{4.8}$$

Proof of (4.8). For $0 < \epsilon$ small enough, dividing (4.7) by $U''(\chi(\kappa))$ and integrating between k and κ^* yields

$$\begin{aligned} &\int_k^{\kappa^*} \frac{U'(\chi(\kappa))}{U''(\chi(\kappa))}(\rho - f'(\kappa))d\kappa + \int_k^{\kappa^*} (f(\kappa^*) - f(\kappa))\chi'(\kappa)d\kappa \\ &= \int_k^{\kappa^*} (\chi(\kappa^*) - \chi(\kappa))\chi'(\kappa)d\kappa = \frac{1}{2}(\chi(\kappa^*) - \chi(k))^2. \end{aligned} \tag{4.9}$$

Let us deal with the first integral in the left hand side of (4.9). Since $f \in W^{2,\infty}_{\text{loc}}$, there exists $\epsilon_0 > 0$ and $C_0 > 0$ such that for all $k \in [\kappa^* - \epsilon_0, \kappa^*]$,

$$\rho - f'(\kappa) = f'(\kappa^*) - f'(\kappa) = \int_\kappa^{\kappa^*} f''(z)dz \geqslant -C_0(\kappa^* - \kappa),$$

thus

$$\int_k^{\kappa^*} \frac{U'(\chi(\kappa))}{U''(\chi(\kappa))}(\rho - f'(\kappa))d\kappa \leqslant -C_0 \int_k^{\kappa^*} \frac{U'(\chi(\kappa))}{U''(\chi(\kappa))}(\kappa^* - \kappa)d\kappa \tag{4.10}$$

Since $U'(\chi(\kappa))/U''(\chi(\kappa))$ admits a negative limit as $\kappa \to \kappa^*$, there exists $C_1 > 0$ such that for all $k \in [\kappa^* - \epsilon_0, \kappa^*]$,

$$\int_k^{\kappa^*} \frac{U'(\chi(\kappa))}{U''(\chi(\kappa))}(\rho - f'(\kappa))d\kappa \leqslant C_1(\kappa^* - k)^2. \tag{4.11}$$

Next, integrating by part the second integral in (4.9) yields

$$\begin{aligned}\int_k^{\kappa^*}(f(\kappa^*)-f(\kappa))\chi'(\kappa)d\kappa &= \int_k^{\kappa^*} f'(\kappa)(\chi(\kappa)-\chi(k))d\kappa \\ &= (\chi(\kappa^*)-\chi(k))\int_k^{\kappa^*} f'(\kappa)\frac{\chi(\kappa)-\chi(k)}{\chi(\kappa^*)-\chi(k)}d\kappa.\end{aligned} \tag{4.12}$$

Setting $J(k)=\int_k^{\kappa^*} f'(\kappa)\frac{\chi(\kappa)-\chi(k)}{\chi(\kappa^*)-\chi(k)}d\kappa$, and using that both f and χ are increasing, we obtain

$$0 \leqslant J(k) \leqslant f(\kappa^*)-f(k).$$

Hence, there exists $\epsilon_1>0$ and $M_1>0$ and such that if

$$0 \leqslant J(k) \leqslant M_1(\kappa^*-k), \qquad \text{for all } k \in [\kappa^*-\epsilon_1, \kappa^*]. \tag{4.13}$$

From (4.9), (4.11) and (4.12), one deduces that for $\epsilon \leqslant \min(\epsilon_0,\epsilon_1)$,

$$(\chi(\kappa^*)-\chi(k))^2 \leqslant 2C_1(\kappa^*-k)^2+2(\chi(\kappa^*)-\chi(k))J(k). \tag{4.14}$$

Elementary algebra yields that for all $k\in[\kappa^*-\epsilon,\kappa^*]$,

$$\begin{aligned}0 \leqslant \chi(\kappa^*)-\chi(k) &\leqslant J(k)+\left(J^2(k)+2C_1(\kappa^*-k)^2\right)^{\frac{1}{2}} \\ &\leqslant \left(M_1+\left(M_1^2+2C_1\right)^{\frac{1}{2}}\right)(\kappa^*-k),\end{aligned} \tag{4.15}$$

where the last inequality is a consequence of (4.13). The bound in (4.8) is proved.

Finally, the definition of κ^* in (2.10) implies that $f(k)-\chi(\kappa^*)=f(k)-f(\kappa^*)=-\rho(\kappa^*-k)+o(\kappa^*-k)$. Therefore, from (4.8), there exists $\epsilon>0$ and $M>0$ such that for all $k\in[\kappa^*-\epsilon,\kappa^*]$,

$$0 \leqslant H_q(k,u'(k)) = f(k)-\chi(k) \leqslant M(\kappa^*-k),$$

which achieves the proof of (4.3). ☐

REMARK 4.1. *Note that under the additional assumption that f is locally uniformly concave, (i.e. for every compact set $K\subset(0,+\infty)$, there exists $\theta>0$ such that $f''(k)\leqslant-\theta$ for all $k\in K$), it can be checked with a similar argument to the one in the proof of Lemma 4.1 that there exists $\epsilon>0$ and $M_1>0$ such that for every $k\in[\kappa^*-\epsilon,\kappa^*+\epsilon]$,*

$$|H_q(k,u'(k))| \geqslant M_1|\kappa^*-k|. \tag{4.16}$$

Consider $k\neq\kappa^*$ such that $|k-\kappa^*|\leqslant\epsilon$; by differentiating (2.7) *at k, we obtain*

$$u''(k)=\frac{u'(k)\,(\rho-f'(k))}{H_q(k,u'(k))}.$$

Using estimate (4.16) *and the regularity of f, we deduce that there exists a constant $M_2>0$ independent of k taken in $[\kappa^*-\epsilon,\kappa^*+\epsilon]$ such that*

$$|u''(k)| \leqslant M_2u'(k).$$

This shows that $u'' \in L^\infty(\kappa^* - \epsilon, \kappa^* + \epsilon)$. Finally, $u \in W^{2,\infty}_{\mathrm{loc}}(0, +\infty)$.

Proof. [Proof of Proposition 2.2] For brevity, we use the notation $b(k) = H_q(k, u'(k))$. If m satisfies (2.11) in the sense of distributions and (2.12), then the weak derivative of bm is $\eta(\cdot, u(\cdot)) - \nu m$, a bounded measure from (2.12) and Assumption 2.3. Hence $bm \in \mathrm{BV}_{\mathrm{loc}}(0, +\infty)$. On the other hand, $1/b \in C^1((0, \kappa^*) \cup (\kappa^*, +\infty))$. Therefore, the restriction of m to $(0, \kappa^*) \cup (\kappa^*, +\infty)$ can be written $(bm)/b$ and identified with a function in $\mathrm{BV}_{\mathrm{loc}}((0, \kappa^*) \cup (\kappa^*, +\infty))$. The Lebesgue decomposition of m is $m = m_{ac} + m_s$; the singular part m_s is supported in $\{\kappa^*\}$, hence $m_s = \lambda \delta_{\kappa^*}$ with $\lambda \geqslant 0$; the regular part m_{ac} can be identified with a nonnegative function in $L^1(0, +\infty)$.
We claim that $\lambda = 0$. To prove this fact, consider a family $(\varphi_\varepsilon)_{\varepsilon > 0}$ such that

- $\varphi_\varepsilon \in C^\infty_c(0, +\infty)$
- $\mathrm{supp}(\varphi_\varepsilon) \subset [\kappa^* - \varepsilon, \kappa^* + \varepsilon]$
- $\varphi_\varepsilon(\kappa^*) = 1$
- φ_ε is non decreasing on $[0, \kappa^*]$, and non increasing in $[\kappa^*, +\infty)$
- $\|\varphi'_\varepsilon\|_\infty \leqslant 2/\varepsilon$

We deduce from (2.11)-(2.12) that for ε small enough,

$$- \int_{\mathbb{R}_+} \varphi'_\varepsilon(k) b(k) dm(k) = -\nu \int_{\mathbb{R}_+} \varphi_\varepsilon(k) dm(k) + \int_{\mathbb{R}_+} \varphi_\varepsilon(k) \eta(k, u(k)) dk.$$

For $\varepsilon \in (0, \kappa^*/2)$, this leads to

$$\begin{aligned} & - \int_{\kappa^* - \varepsilon}^{\kappa^* + \varepsilon} \varphi'_\varepsilon(k) b(k) m_{ac}(k) dk \\ = & - \nu \int_{\kappa^* - \varepsilon}^{\kappa^* + \varepsilon} \varphi_\varepsilon(k) m_{ac}(k) dk + \int_{\kappa^* - \varepsilon}^{\kappa^* + \varepsilon} \varphi_\varepsilon(k) \eta(k, u(k)) dk - \nu \lambda, \end{aligned}$$

because $b(\kappa^*) = 0$. The construction of φ_ε and (4.3)-(4.4) ensure that

$$\sup_{k \in [\kappa^* - \varepsilon, \kappa^* + \varepsilon]} |\varphi'_\varepsilon(k) b(k)| \leqslant 2M.$$

This yields

$$0 \leqslant \nu \lambda \leqslant 2M \int_{k^* - \varepsilon}^{k^* + \varepsilon} m_{ac}(k) dk + \int_{\kappa^* - \varepsilon}^{\kappa^* + \varepsilon} \varphi_\varepsilon(k) \eta(k, u(k)) dk.$$

Letting $\varepsilon \to 0$, we obtain that $\lambda = 0$ by applying Lebesgue dominated convergence theorem. The claim is proved.
Therefore, $m \in L^1(0, +\infty)$, and (4.3) implies that $bm \in W^{1,1}_{\mathrm{loc}}(0, +\infty)$, and that $0 \leqslant m \in L^1(0, +\infty) \cap C^1((0, \kappa^*) \cup (\kappa^*, +\infty))$.
Integrating equation (2.11) over the intervals $(0, \kappa^*)$ and $(\kappa^*, +\infty)$, we see that

$$b(k)m(k) =$$
$$\begin{cases} \displaystyle \int_0^k \eta(\kappa, u(\kappa)) \exp\left(- \int_\kappa^k \frac{\nu}{b(z)} dz \right) d\kappa + A \exp\left(- \int_{\frac{\kappa^*}{2}}^k \frac{\nu}{b(z)} dz \right), & \text{if } 0 < k < \kappa^*, \\ \displaystyle - \int_k^\infty \eta(\kappa, u(\kappa)) \exp\left(\int_k^\kappa \frac{\nu}{b(z)} dz \right) d\kappa + B \exp\left(\int_k^{\frac{3\kappa^*}{2}} \frac{\nu}{b(z)} dz \right), & \text{if } k > \kappa^*, \end{cases}$$

for two real numbers A and B. But, from Lemma 4.1, we see that a necessary condition for the integrability of m is that $A = B = 0$. Imposing $A = B = 0$, we see that m is a nonnegative function. It remains to check (2.12). Set $I_1 = \int_0^{\kappa^*} m(k)dk$ and $I_2 = \int_{\kappa^*}^{+\infty} m(k)dk$.
Focusing on I_1,

$$\begin{aligned} I_1 &= \int_0^{\kappa^*} \frac{1}{b(k)} \int_0^k \eta(\kappa, u(\kappa)) \exp\left(-\int_\kappa^k \frac{\nu}{b(z)} dz\right) d\kappa dk, \\ &= \int_0^{\kappa^*} \eta(\kappa, , u(\kappa)) \int_\kappa^{\kappa^*} \frac{1}{b(k)} \exp\left(-\int_\kappa^k \frac{\nu}{b(z)} dz\right) dk d\kappa, \\ &= \frac{1}{\nu} \int_0^{\kappa^*} \eta(\kappa, u(\kappa)) d\kappa. \end{aligned} \tag{4.17}$$

The second line in (4.17) is obtained using the non negativity of the integrand and Tonelli's theorem. The third line in (4.17) comes from the fact that $\int_\kappa^{\kappa^*} \frac{\nu}{b(z)} dz = +\infty$, which is a consequence of Lemma 4.1.
It can be proved in the same way that

$$I_2 = \frac{1}{\nu} \int_{\kappa^*}^{+\infty} \eta(\kappa, u(\kappa)) d\kappa.$$

Hence $\nu(I_1 + I_2) = \int_{\mathbb{R}_+} \eta(\kappa, u(\kappa)) d\kappa$, and m given by (2.13) is the unique solution of (2.11)-(2.12). □

5. Equilibrium. This paragraph is devoted to existence of equilibria.

5.1. Stability results for (2.7)-(2.9)**.** Lemma 5.1. *Under Assumptions 2.1 and 2.2, the value function $-u$ is monotone with respect to w, i.e. for every $w, \tilde{w} \in (0, +\infty)^d$,*

$$w \leqslant \tilde{w} \qquad \Rightarrow \qquad u(\cdot, w) \geqslant u(\cdot, \tilde{w}).$$

Proof. Assume $w \leqslant \tilde{w}$ and consider an admissible trajectory associated with the vector of prices $\tilde{w}$: it satisfies $\frac{dk}{dt}(t) = f(k(t), \tilde{w}) - c(t)$ with $k(0) = k_0$. The differential equation also reads: $\frac{dk}{dt}(t) = f(k(t), w) - (c(t) + f(k(t), w) - f(k(t), \tilde{w}))$, and $c(t) + f(k(t), w) - f(k(t), \tilde{w}) \geqslant c(t) \geqslant 0$, which can be used as a control. This yields that

$$u(k_0, w) \geqslant \int_0^\infty U\left(c(t) + f(k(t), w) - f(k(t), \tilde{w})\right) e^{-\rho t} dt \geqslant \int_0^\infty U(c(t)) e^{-\rho t} dt.$$

Taking the supremum on all admissible trajectories associated with $\tilde{w}$, we deduce that for all $k_0 > 0$, $u(k_0, w) \geqslant u(k_0, \tilde{w})$. □

Lemma 5.2. *Under Assumption 2.2, the map $(0, +\infty)^d \ni w \mapsto \kappa^*(w) \in (0, +\infty)$ defined in* (2.10) *is continuous.*

Proof. Consider a sequence $(w_n)_{n \in \mathbb{N}}$, $w_n \in (0, +\infty)^d$, such that w_n tends to $w \in (0, +\infty)^d$ as $n \to +\infty$.
We first claim that $\kappa^*(w_n)$ remains in a compact subset of $(0, +\infty)$. We proceed by contradiction:

- Assume first that up to the extraction of a subsequence, $\kappa^*(w_n)$ tends to $+\infty$ as $n \to +\infty$. Hence, for any $k > 0$, there exists $N > 0$ such that if $n \geqslant N$, then $\frac{\partial f}{\partial k}(k, w_n) > \rho$. Passing to the limit using the C^1 regularity of f (see Assumption 2.2), we get that $\frac{\partial f}{\partial k}(k, w) \geqslant \rho$ for all $k > 0$. But $k \mapsto f(k, w)$ is strictly concave: Hence, $\frac{\partial f}{\partial k}(k, w) > \rho$ for all $k > 0$. This contradicts point 2.iii in Assumption 2.2 (see also Remark 2.2).
- Assume that up to the extraction of a subsequence, $\kappa^*(w_n)$ tends to 0 as $n \to +\infty$: arguing as above, this implies that $\frac{\partial f}{\partial k}(k, w) < \rho$ for all $k > 0$. This contradicts point 2.ii in Assumption 2.2.

The claim is proved.
Possibly after the extraction of a subsequence, $\kappa^*(w_n)$ tends to a positive limit $\tilde{\kappa}$. It is easy to deduce from Assumption 2.2 that $\frac{\partial f}{\partial k}(\tilde{\kappa}, w) = \rho$. Therefore $\tilde{\kappa} = \kappa^*(w)$, and the uniqueness of the cluster point implies that the whole sequence $\kappa^*(w_n)$ tends to $\kappa^*(w)$. This achieves the proof. □

LEMMA 5.3 (Continuity of $w \mapsto u(\cdot, w)$). *Let $(w_n)_{n\in\mathbb{N}}$, $w_n \in (0, +\infty)^d$, be a sequence converging to $w \in (0, +\infty)^d$ as $n \to \infty$. Then, under Assumptions 2.1 and 2.2,*

$$u(\cdot, w_n) \to u(\cdot, w)$$

in $C^1(K)$ for every compact subset K of $(0, +\infty)$.

Proof. We may assume without loss of generality that there exist two vectors $\underline{w}, \overline{w} \in (0, +\infty)^d$ such that, for all $n \geqslant 0$,

$$\underline{w} \leqslant w_n \leqslant \overline{w}.$$

From Lemma 5.1, the following inequalities hold for all $n \geqslant 0$:

$$u(\cdot, \overline{w}) \leqslant u(\cdot, w_n) \leqslant u(\cdot, \underline{w}).$$

Using (2.7) and the coercivity of $q \mapsto H(k, q, w_n)$ uniform w.r.t. n and $k \in K$, where K is a compact subset of $(0, +\infty)$, we see that $\frac{\partial u}{\partial k}(k, w_n)$ is bounded uniformly in n and $k \in K$. Moreover, if $\lim_{c\to+\infty} U(c) = +\infty$, then $\frac{\partial u}{\partial k}(\cdot, w_n)$ is bounded uniformly away from 0 w.r.t. n and $k \in K$.
Since $(u(\cdot, w_n))_{n\in\mathbb{N}}$ is a sequence of concave functions uniformly bounded on every compact subset of $(0, +\infty)$, there exists a continuous and concave function $v : (0, +\infty) \to \mathbb{R}$ such that, after the extraction of a subsequence,

- $u(\cdot, w_n) \to v$ locally uniformly in $(0, +\infty)$
- $\frac{\partial u}{\partial k}(\cdot, w_n) \to v'$ almost everywhere in $(0, +\infty)$.

On the other hand, from Lemma 5.2, there exist $\underline{\kappa} > 0$ and $\overline{\kappa} > \underline{\kappa}$ such that

$$\underline{\kappa} < \min_{\underline{w} \leqslant w \leqslant \overline{w}} \kappa^*(w) \leqslant \max_{\underline{w} \leqslant w \leqslant \overline{w}} \kappa^*(w) < \overline{\kappa}.$$

Take any compact interval $[a, b]$ such that $0 < a < \underline{\kappa}$ and $\overline{\kappa} < b$.

The functions $u(\cdot, w_n)$ are uniformly Lipschitz viscosity solutions (with $\frac{\partial u}{\partial k}(\cdot, w_n)$ bounded away from 0 if $\lim_{c\to+\infty} U(c) = +\infty$) of (2.7) (with $w = w_n$) on (a, b) with state constrained boundary conditions at a and b. From the continuity of H on $[a, b] \times (0, +\infty)^d \times (0, +\infty)^d$, the uniform bounds on $\frac{\partial u}{\partial k}(\cdot, w_n)$ stated above and the uniform convergence of $(u(\cdot, w_n))_{n\in\mathbb{N}}$ towards v on $[a, b]$, stability results on viscosity solutions, see e.g. [6] can be used and yield that v is a viscosity solution of

$$\rho v(k) = H(k, v'(k), w),$$

on (a,b), with state constrained boundary conditions at $k=a$ and $k=b$. Note that the eventuality that $H(k,q,w)\to+\infty$ as $q\to 0$ does not imply any difficulty, because in this case, $\frac{\partial u}{\partial k}(\cdot,w_n)$ is uniformly bounded away from 0. From this observation, we can also use well-known results on the uniqueness of state constrained solutions of the Hamilton-Jacobi equation, see e.g. [6], and find that $v=u(\cdot,w)$.

In fact, the convergence holds locally in C^1. We know that

- $u(\cdot,w_n)$ tends to $u(\cdot,w)$ uniformly in $[a,b]$
- there exists a measurable subset E of $[a,b]$, such that the Lebesgue measure of $[a,b]\backslash E$ is zero and that $\frac{\partial u}{\partial k}(\cdot,w_n)$ tends to $\frac{\partial u}{\partial k}(\cdot,w)$ pointwise in E.

Note that after slightly modifying a or b if necessary, we can always assume that $a\in E$ and $b\in E$. A variant of Dini's first theorem yields that the convergence of $\frac{\partial u}{\partial k}(\cdot,w_n)$ is in fact uniform in $[a,b]$: for completeness, the proof is given in what follows.
The function $\frac{\partial u}{\partial k}(\cdot,w)$ is continuous, thus uniformly continuous on $[a,b]$; hence, given $\epsilon>0$, it is possible to choose $\delta>0$ small enough such that

$$|k-k'|\leqslant\delta \quad\Rightarrow\quad \left|\frac{\partial u}{\partial k}(k,w)-\frac{\partial u}{\partial k}(k',w)\right|<\frac{\varepsilon}{2},\qquad \forall k,k'\in[a,b].$$

For such a choice of $\delta>0$, it is possible to define a finite subdivision $(\sigma_i)_{i\in\{0,\dots,I\}}$ of $[a,b]$ such that

- for every $i\in\{0,\dots,I\}$, $\sigma_i\in E$.
- for any $i\in\{0,\dots,I-1\}$, $0<\sigma_{i+1}-\sigma_i<\delta$.

On the other hand, for any $k\in[a,b]$, there exists $i_0\in\{0,\dots I-1\}$ such that $\sigma_{i_0}\leqslant k\leqslant\sigma_{i_0+1}$. Then the concavity of u with respect to k yields

$$\begin{aligned}\frac{\partial u}{\partial k}(k,w_n)-\frac{\partial u}{\partial k}(k,w)&\leqslant\frac{\partial u}{\partial k}(\sigma_{i_0},w_n)-\frac{\partial u}{\partial k}(\sigma_{i_0+1},w)\\&=\frac{\partial u}{\partial k}(\sigma_{i_0},w_n)-\frac{\partial u}{\partial k}(\sigma_{i_0},w)+\frac{\partial u}{\partial k}(\sigma_{i_0},w)-\frac{\partial u}{\partial k}(\sigma_{i_0+1},w).\end{aligned}$$

Taking $N\in\mathbb{N}$ large enough such that for every $n\geqslant N$,

$$\max_{0\leqslant i\leqslant I}\left|\frac{\partial u}{\partial k}(\sigma_i,w_n)-\frac{\partial u}{\partial k}u(\sigma_i,w)\right|<\frac{\varepsilon}{2}$$

yields that

$$\frac{\partial u}{\partial k}(k,w_n)-\frac{\partial u}{\partial k}u(k,w)<\varepsilon,\quad\forall n\geqslant N.$$

A similar argument can be used to bound $\frac{\partial u}{\partial k}(k,w_n)-\frac{\partial u}{\partial k}(k,w)$ from below. Finally, for any $\varepsilon>0$ there exists $N>0$ such that

$$\sup_{k\in[a,b]}\left|\frac{\partial u}{\partial k}(k,w_n)-\frac{\partial u}{\partial k}(k,w)\right|<\varepsilon,\qquad\forall n\geqslant N.$$

This achieves the proof.
□

5.2. Existence of equilibria. *Proof.* [Proof of Theorem 2.3] Recall that Φ and g are respectively defined in Assumption 2.4 and formula (2.16). Let ϵ be the constant appearing in Assumption 2.5. There exist two constants $0<\underline{\kappa}\leqslant\overline{\kappa}<+\infty$ such that for all $w \in [\epsilon, 1/\epsilon]^d$, $\underline{\kappa} < \kappa^*(w) < \overline{\kappa}$. Hence, $m(\cdot, w)$ is supported in the compact inteval $J = \text{conv}\left([\underline{\kappa}, \overline{\kappa}] \cap \text{support}(\hat{\eta})\right)$.

We claim that the map $w \mapsto m(\cdot, w)$ is continuous from $[\epsilon, 1/\epsilon]^d$ to the set of probability measures supported in J. Indeed, let $(w_n)_{n\in\mathbb{N}}$, $w_n \in [\epsilon, 1/\epsilon]^d$, be a sequence converging to w as $n \to +\infty$. From Lemma 5.3, $u(\cdot, w_n) \to u(\cdot, w)$ in $C^1(K)$ for any compact subset K of $(0, +\infty)$. The probability measures $m(\cdot, w_n)$ are all supported in J. Hence, the sequence $m(\cdot, w_n)$ has a cluster point μ in the weak $*$ topology. Let us prove that $\mu = m(\cdot, w)$: for any test function $\phi(\cdot) \in C_c^\infty(0, +\infty)$,

$$-\int_0^{+\infty} \phi'(k) b(k, w_n) m(k, w_n) dk = \int_0^{+\infty} \phi(k)\eta(k) dk - \nu \int_0^{+\infty} \varphi(k) m(k, w_n) dk.$$

where b is given by (2.14).

The right-hand side converges to $\int_0^{+\infty} \phi(k)\eta(k)dk - \nu \int_0^{+\infty} \phi(k)\mu(k)dk$. On the other hand, the C^1 convergence of $u(\cdot, w_n)$ to $u(\cdot, w)$ on every compact subset of $(0, +\infty)$ implies the uniform convergence of $H_q(\cdot, \frac{\partial u}{\partial k}(\cdot, w_n), w_n)$ to $H_q\left(\cdot, \frac{\partial u}{\partial k}(\cdot, w), w\right)$ in J. We deduce that

$$\int_0^\infty \phi'(k) H_q\left(k, \frac{\partial u}{\partial k}(k, w_n), w_n\right) m(k, w_n) dk \to \int_0^\infty \phi'(k) H_q\left(k, \frac{\partial u}{\partial k}(k, w), w\right) \mu(k) dk.$$

Therefore $\mu = m(\cdot, w)$ and the whole sequence $m(\cdot, w_n)$ weakly $*$ converges to $m(\cdot, w)$ as $n \to \infty$. The map $w \mapsto m(\cdot, w)$ is continuous on $[\epsilon, 1/\epsilon]^d$.

For $\lambda \in [0, 1]$, we then consider the map T_λ defined on $[\epsilon, 1/\epsilon]^d$ by

$$T_\lambda(w) = \arg\min \left\{ \Phi(\cdot) + \int_0^\infty g(k, \cdot)\Big((1-\lambda) d\hat{\eta}(k) + \lambda dm(k, w)\Big) \right\}, \tag{5.1}$$

where the function g has been defined in (2.16), (recall that $k \mapsto g(k)$ is convex). From the observation made above on $m(\cdot, w)$ and from Assumption 2.4, the function to be minimized is continuous, strictly convex and coercive on $[0, +\infty)^d$; hence $T_\lambda(w)$ is well defined. Moreover, $\|T_\lambda(w)\|_\infty$ is bounded uniformly in $w \in [\epsilon, 1/\epsilon]^d$.
Let w_n and λ_n be two sequences taking their values respectively in $[\epsilon, 1/\epsilon]^d$ and in $[0, 1]$; assume that w_n tends to w and that λ_n tends to λ. The sequence $T_{\lambda_n}(w_n)$ takes its values in a compact; hence, up to the extraction of a subsequence, we may assume that $T_{\lambda_n}(w_n)$ converges to some $\tilde{w}$. Since $m(\cdot, w_n)$ weakly $*$ converges to $m(\cdot, w)$, it is easy to check that $\tilde{w} = T_\lambda(w)$ and that the whole sequence $T_{\lambda_n}(w_n)$ converges. Hence, the map $(\lambda, w) \mapsto T_\lambda(w)$ is continuous.

For $\lambda \in [0, 1]$, we consider the equation: find $w \in [\epsilon, 1/\epsilon]^d$ such that $w - T_\lambda(w) = 0$, which we write $\chi(w, \lambda) = 0$. We now aim at applying Brouwer degree theory to χ.

First, setting $t_0 = \arg\min \left\{\Phi(\cdot) + \int_0^\infty g(k, \cdot) d\hat{\eta}(k)\right\}$ which does not depend on w, the equation $\chi(w, 0) = 0$ writes $w = t_0 \in (\epsilon, 1/\epsilon)^d$. Therefore,

$$\deg\left(\chi(\cdot, 0), (\epsilon, 1/\epsilon)^d, 0_{\mathbb{R}^d}\right) = 1. \tag{5.2}$$

Second, for all $\lambda \in [0, 1]$, we know from Assumption 2.5 that the equation $w - T_\lambda(w) = 0$ has no solution on the boundary of $[\epsilon, 1/\epsilon]^d$.

From the two observations above, we see that for all $\lambda \in [0, 1]$,

$$\deg\left(\chi(\cdot, \lambda), (\epsilon, 1/\epsilon)^d, 0_{\mathbb{R}^d}\right) = 1. \tag{5.3}$$

We deduce that there exists $w^* \in (\epsilon, 1/\epsilon)^d$ such that

$$w^* = \arg\min \left\{ \Phi(\cdot) + \int_0^\infty g(k,\cdot) dm(k, w^*) \right\}.$$

Writing the first order necessary optimality conditions associated with this minimization problem, we see that w^* satisfies (2.15). ◻

REMARK 5.1. *We have actually proved more than the existence of an equilibrium, namely that deg*$(\chi, (\epsilon, 1/\epsilon)^d, 0) = 1$.

5.3. Assumption 2.5 holds in the examples of Subsection 2.4.

5.3.1. The Cobb-Douglas production function. PROPOSITION 5.4. *Assumption 2.5 holds with the Cobb-Douglas production function described in Subsection 2.4 .*

Proof. From (2.19), we deduce that for two positive constants c_1 and c_2

$$g(k,w) = c_1 k^{\frac{\alpha}{1-|\beta|}} G_\beta(w) \quad \text{and} \quad \kappa^*(w) = c_2 \left(G_\beta(w)\right)^{\frac{1-|\beta|}{1-\alpha-|\beta|}}, \tag{5.4}$$

where

$$G_\beta(w) = \prod_{i=1}^d w_i^{-\frac{\beta_i}{1-|\beta|}}.$$

Setting

$$M_\lambda(w) = \left(\lambda c_1 \int_0^\infty k^{\frac{\alpha}{1-|\beta|}} dm(k,w) + (1-\lambda) M_0 \right) \qquad \text{with} \quad M_0 = c_1 \int_0^\infty k^{\frac{\alpha}{1-|\beta|}} d\hat{\eta}(k),$$

(2.17) becomes

$$\Phi(w) + M_\lambda(w) G_\beta(w) \leqslant \Phi(\mathbb{1}) + M_\lambda(w). \tag{5.5}$$

Since $\Phi(w) \geqslant 0$, (5.5) implies that $G_\beta(w) \leqslant 1 + \Phi(\mathbb{1})/M_\lambda(w)$. On the other hand, (5.4) yields

$$M_\lambda(w) \geqslant c_1 \lambda \min\left(\underline{a}, c_2 \left(G_\beta(w)\right)^{\frac{1-|\beta|}{1-\alpha-|\beta|}}\right)^{\frac{\alpha}{1-|\beta|}} + (1-\lambda) M_0, \tag{5.6}$$

where $\underline{a}$ is the minimal value in the support of $\hat{\eta}$. Combining the latter two estimates yields

$$G_\beta(w) \leqslant 1 + \frac{\Phi(\mathbb{1})}{c_1 \lambda \min\left(\underline{a}, c_2 \left(G_\beta(w)\right)^{\frac{1-|\beta|}{1-\alpha-|\beta|}}\right)^{\frac{\alpha}{1-|\beta|}} + (1-\lambda) M_0}. \tag{5.7}$$

It is easy to deduce from (5.7) that $G_\beta(w) < c_3$, for a positive constant c_3 independent of w.

If $\overline{a}$ is the maximal value in the support of $\hat{\eta}$, this implies that

$$\begin{aligned} M_\lambda(w) &\leqslant c_1 \lambda \max\left(\overline{a}, c_2 \left(G_\beta(w)\right)^{\frac{1-|\beta|}{1-\alpha-|\beta|}}\right)^{\frac{\alpha}{1-|\beta|}} + (1-\lambda) M_0 \\ &\leqslant c_1 \lambda \max\left(\overline{a}, c_2 c_3^{\frac{1-|\beta|}{1-\alpha-|\beta|}}\right)^{\frac{\alpha}{1-|\beta|}} + (1-\lambda) M_0 \\ &= c_4, \end{aligned} \tag{5.8}$$

where c_4 is a positive constant. We deduce from this and (5.5) that

$$\Phi(w) \leqslant \Phi(\mathbb{1}) + M_\lambda(w) \leqslant \Phi(\mathbb{1}) + c_4. \tag{5.9}$$

From the coercivity of Φ, this yields that $\max_i w_i < c_5$, for a positive constant c_5. Then $G_\beta(w) < c_3$ implies that $\min_i w_i > \epsilon$, where ϵ is a positive constant which can be obtained from the exponents β_i and the constants c_3 and c_5. Finally, taking a smaller value of ϵ if necessary, we get (2.18). □

5.3.2. Constant elasticity of substitution. PROPOSITION 5.5. *Assumption 2.5 holds with the example of the production function with the constant elasticity of substitution described in Subsection 2.4 .*

Proof. Combining (2.23) and (2.26) implies that

$$\gamma = \frac{\delta+\rho}{\alpha}\left((\kappa^*(w))^\alpha + \sum_{j=1}^d \left(\frac{\lambda\beta_j}{w_j}\right)^{\frac{\beta_j}{1-\beta_j}}\right)^{1-\gamma} (\kappa^*(w))^{1-\alpha}.$$

Since $\gamma \in (0,1)$, this yields

$$\frac{\delta+\rho}{\alpha}(\kappa^*(w))^{1-\alpha+\alpha(1-\gamma)} \leqslant \gamma. \tag{5.10}$$

Hence $\kappa^*(w)$ is bounded from above by a positive constant independent of w. From this information and the coercivity of Φ, we proceed as for the Cobb-Douglas function and see that there exists a positive constant c_1 such that (2.17) implies that $\|w\|_\infty < c_1$.

Next, we claim that

$$\lim_{\left\{\begin{array}{l}\min_{i=1,\dots,d} w_i \to 0,\\ \|w\|_\infty \leqslant c_1\end{array}\right.} g(0,w) = +\infty. \tag{5.11}$$

Since $g(\cdot,w)$ is non decreasing, we deduce from (5.11) that there exists a constant $\epsilon > 0$ independent of λ such that (2.17) implies $\min_i w_i > \epsilon$ and taking a smaller value of ϵ if necessary, we get (2.18).

We are left with proving (5.11): we know that

$$g(k,w) \geqslant g(0,w) = \sup_\ell \left(\sum_i \ell_i^{\beta_i}\right)^\gamma - w\cdot\ell.$$

A competitor can be chosen by taking $\tilde{\ell}_i = w_i^{-\frac{b}{\beta_i}}$ where $b = \min_i \beta_i/2$. Therefore $g(k,w) \geqslant \left(\sum_i w_i^{-b}\right)^\gamma - \sum_i w_i^{1-\frac{b}{\beta_i}}$. The first term tends to $+\infty$ if $\min_i w_i \to 0$, while the second term is bounded since $\|w\|_\infty \leqslant c_1$. □

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