A mean field model for the interactions between firms on the markets of their inputs : numerical simulations

Quentin Petit

Contents

1 Introduction

This work comes from Chapter 7 in the manuscipt [6]. We aim at solving numerically the following MFG system

$$\begin{aligned} \rho u(k) &= H\left(k, u'(k)\right), & (1)\\ \frac{d}{dk}\left(D_q H\left(\cdot, u'(\cdot)\right) m(\cdot)\right)(k) &= \eta(k) - \nu m(k), & (2)\\ S(w) &= -\int_0^{+\infty} D_w f(k) dm(k), & (3) \end{aligned}$$

completed with the following conditions:

$$\begin{aligned} D_q H(0, u'(0)) &\geq 0, & (4)\\ 1 &= \int_0^{+\infty} dm(k). & (5) \end{aligned}$$

where the Hamiltonian $H : [0, +\infty) \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is

$$H(k, q) = \sup_{c \geq 0} \{U(c) - cq\} + f(k)q, \quad \forall (k, q) \in (0, +\infty)^2. \tag{6}$$

We assume that the net output $f : [0, +\infty) \times (0, +\infty)^d \to \mathbb{R}$ and the labour supply $S : (0, +\infty)^d \to (0, +\infty)^d$ are given and essentially satisfy the assumptions [2]. The system (1)-(5) characterizes an equilibrium where equation (1) completed with (4) gives the strategy of the firms, (2) completed with (5) gives the capital distribution, and (3) corresponds to the market clearing conditions. Note that for simplicity η only depends k but the algorithm can be extended to cover the general case developed in [2].

For a given vector of wages $w \in (0, +\infty)^d$, the approximation used is inspired from the one proposed by Achdou and Capuzzo-Dolcetta in [1]. Once the two equations are solved we can approximate the integral in (2) and compute $S(w) + \int_0^{+\infty} D_w f(k) dm(k)$. We then aim at finding a zero of the function

$$(0, +\infty)^d \ni w \mapsto S(w) + \int_0^{+\infty} D_w f(k) dm(k)$$

in order to exhibit a solution of the problem.
We use this method to make simulations. We fix some parameters from the economic literature [5], while the others are fixed in order to have solutions which well represent the data we get from the Conseil Supérieur de l'Audiovisuel and from Insee [3, 4].

2 The finite difference operators

2.1 The scheme

In this paragraph, we present the approximation of (1)-(5). Let us fix $w \in (0, +\infty)^d$. As noted in Remark 2.5 of [2], the density of probability m has a compact support. An upper-bound is $K = \max(\operatorname{supp} \eta, k^*(w))$ where $k^*(w)$ is the unique solution of

$$\frac{\partial f}{\partial k} f(k, w) = \rho. \tag{7}$$

For $\epsilon > 0$, we set $\Gamma = (0, K + \epsilon]$ and let Γ_h be a uniform grid on Γ with mesh step h (assuming that $1/h$ is an integer N_h). Let k_i denotes a generic point in Γ_h. The values of u and m at k_i will respectively be approximated by U_i and M_i. We introduce the finite difference operator:

$$(D^+ U)_i = \frac{U_{i+1} - U_i}{h}, \quad (i = 1, ..., N_h - 1).$$

Given a level of capital $k \in (0, +\infty)$, we can split the Hamiltonian into its non decreasing part and its non increasing part with respect to q : $H(k, \cdot) = H^{\downarrow}(k, \cdot) + H^{\uparrow}(k, \cdot) - \min_{q \in (0, +\infty)} H(k, q)$. If there is no increasing part with respect to q, then $H(k, \cdot) = H^{\downarrow}(k, \cdot)$. The approximation of (1) is therefore

$$\rho U_i = \tilde{G}_i(k, U) \quad \forall i = 1, ..., N_h, \tag{8}$$

with

$$\tilde{G}_i(k, U) = H^{\downarrow}\left(k_i, (D^+ U)_{i-1}\right) + H^{\uparrow}\left(k_i, (D^+ U)_i\right) - a(k_i),$$

where

$$a(k_i) = \begin{cases} 0, & \text{if } H^{\uparrow}(k_i, \cdot) \equiv 0, \\ \min_{q\in(0,+\infty)} H(k_i, q), & \text{otherwise,} \end{cases}$$

and where $(D^+U)_0$ and $(D^+U)_{N_h}$ are arbitrarily chosen to ensure that $H^{\downarrow}(k_1, (D^+U)_0) = 0$ and $H^{\uparrow}(k_{N_h}, (D^+U)_{N_h}) = 0$. Indeed, for every $k \in (0, k^*(w))$, $D_qH(k, u'(k))$ must be positive. Therefore, in a neighbourhood of 0, the contribution of the Hamiltonian comes from its increasing part. Similarly, near $K + \epsilon$ the contribution of the Hamiltonian comes from its decreasing part. In order to introduce the approximation used for the continuity equation (2), let us differentiate $\tilde{G}_i$ with respect to U:

$$\begin{aligned} D_U\tilde{G}_i(k, U)V = &- D_qH^{\downarrow}\left(k_i, (D^+U)_{i-1}\right) V_{i-1}/h \\ &+ \left\{D_qH^{\downarrow}\left(k_i, (D^+U)_{i-1}\right) - D_qH^{\uparrow}\left(k_i, (D^+U)_i\right)\right\} V_i/h \\ &+ D_qH^{\uparrow}\left(k_i, (D^+U)_i\right) V_{i+1}/h \end{aligned} \tag{9}$$

for $i = 2, ..., N_h - 1$. In the case $i = 1$ and $i = N_h$ we obtain:

$$D_U\tilde{G}_0(k, U)V = -D_qH^{\uparrow}\left(k_1, (D^+U)_1\right) V_1/h + D_qH^{\uparrow}\left(k_1, (D^+U)_1\right) V_2/h,$$

and

$$D_U\tilde{G}_{N_h}(k_i, U)V = D_qH^{\downarrow}\left(k_{N_h}, (D^+U)_{N_h-1}\right) V_{N_h-1}/h + D_qH^{\downarrow}\left(k_{N_h}, (D^+U)_{N_h-1}\right) V_{N_h}/h.$$

We can summarize the N_h previous lines as $D_U\tilde{G}(k, U)V$. Let us consider Σ the vector in $\mathbb{R}^{N_h}$ such that for every $i = 1, ..., N_h$, $\Sigma_i = \eta(k_i)$. The approximation of the continuity equation (2) is given by

$$(I_d\nu + D_q\tilde{H}(k, U)^T)M = \Sigma. \tag{10}$$

Given $M \in \mathbb{R}^{N_h}$ the solution of (10), we approximate the integral in (3) as follows:

$$-\int_0^{+\infty} D_wf(k)dm(k) \simeq -h\sum_{i=1}^{N_h} D_wf(k_i)M_i. \tag{11}$$

2.2 The algorithm

2.2.1 Solution of the HJ equation

Given a vector of wages $w \in (0, +\infty)^d$, the problem defined in (8) is non linear. Therefore, the Newton's method is used to compute the unique solution U.
Let us introduce $G : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ given for every $V \in R^{N_h}$ by

$$G_i(V) = \rho V_i - \tilde{G}_i(k, V), \quad \forall i = 1, ..., N_h,$$

where the function a has been previously defined. Given an arbitrary initial guess $U^0 \in \mathbb{R}^{N_h}$, for every n, we compute U^{n+1} from U^n by the Newton's iteration:

$$DG(U^n)(U^{n+1} - U^n) = -G(U^n).$$

This sequence converges towards the zero of G.

2.2.2 Solution of the problem

Once the solution of (8) is computed, we solve the system of linear equations (10) and obtain M. This allows us to determinate the residual

$$\Lambda(w) = S(w) + h\sum_{i=1}^{N_h} D_w f(k_i) M_i.$$

When $d = 1$, we use the secant method in order to solve the equation $\Lambda(\omega) = 0$. When $d \geq 2$, we use the so called "good" Broyden's iterations. For completeness we recall that iterations of the secant method are of the form

$$w^{n+1} = w^n - \frac{w^n - w^{n-1}}{\Lambda(w^n) - \Lambda(w^{n-1})}\Lambda(w^n).$$

The "good" Broyden's method consists in the iterations:

$$B^n s^n = -\Lambda(w^n),$$

where $s^n = w^n - w^{n-1}$ and

$$B^n = B^{n-1} + \frac{r^n (s^n)^T}{(s^n)^T s^n}$$

with $r_n = \Lambda(w^n) - \Lambda(w^{n-1})$. Note that both methods need two initial guesses. For the secant method, we simply choose two points w^0, w^1 in $(0, +\infty)$. For the "good" Broyden's method, we choose one point w^0 in $(0, +\infty)^d$ and we specify $B^0 = I_d$.

3 Numerical simulations

The numerical simulations reported below deal with the sector of the audiovisual production and distribution. The data come from the Conseil Supérieur de l'Audiovisuel, see [3], and Insee the French institute for statistics, see [4]. We also use data coming from the economic literature, see [5]. The data from Insee are summarized in Table 1 below.

Annual production / Nb of firms (in 10^4 Euro)	66.4
Total payroll / Nb of firms (in 10^4 Euro)	47.1
Nb of Employees / Nb of firms	5.81
Annual production / Nb of Employees (in 10^4 Euro)	11.4
Total payroll / Nb of Employees (in 10^4 Euro)	8.11

Table 1: Data from Insee.

3.1 Solution of (1)-(5) applied to the audiovisual, publishing and distribution sector

We make two tests in this paragraph. In Test 1, the only factor of production of firms is the workforce. We are able to find a numerical solution in this case. Then, we run Test 2 where we link the labour market with rental market for professionals, and see how the equilibrium is impacted. We also compare the speed of convergence of the algorithms used for these simulations.

3.1.1 Choice of models and parameters

We assume that the production function $F : [0, +\infty) \times [0, +\infty)^d \to [0, +\infty)$ is a Cobb-Douglas function:

$$F(k, \ell) = Ak^{\alpha}\ell^{\beta}, \quad \forall (k, \ell) \in [0, +\infty) \times [0, +\infty)^d.$$

Then, if labour is a control, the elasticity of the production with respect to the variations of the total payroll is given by the following quantity

$$\frac{\text{Total payroll}}{\text{Annual production}} \simeq \frac{8.11}{11.4} = 0.710.$$

From [5], we set the elasticity with respect to capital to $\alpha = 0.21$. We also fix the depreciation rate $\delta = 0.07$, i.e. within a year, firms lose 7% percent of their capital.
The report [3] states that, in 2018 around 26% of the firms in the audiovisual sector were less than three year old. Assuming that the death of firms follows an exponential law, it gives a rate of death of 0.10 in this sector of activity. We retain $\nu = 0.1$ for the whole sector of "Audiovisual, publishing and distribution".
We set the discount factor to $\rho = 0.1$.
We assume that the labour supply is given by a logistic function, i.e. $S_{labour} : [0, +\infty) \to [0, +\infty)$ is given for every $w \in [0, +\infty)$ by

$$S_{labour}(w) = \frac{K}{(1 + e^{-r(w-\mu)})},$$

with $K = 6.5$, $r = 2 \times 10^{-4}$, and $\mu = 7 \times 10^4$ (see Figure 1) and that the instantaneous utility function is a logarithm, i.e.

$$U(c) = \ln(c), \quad \forall c \in (0, +\infty).$$

Moreover, we model the entries of firms by a Gaussian function, times the rate ν, centred in 30×10^4, with a standard deviation of 9×10^4 (see Figure 1).

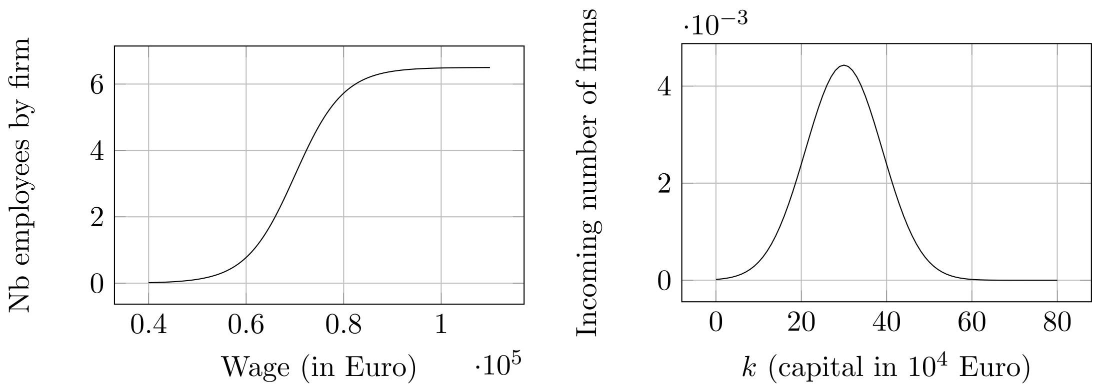


Figure 1: Labour supply and the source.

In Test 2, we also need to model the rental market for professionals. We assume that the workspace supply is given by a logistic function i.e. $S_{workspace} : [0, +\infty) \to [0, +\infty)$

$$S_{workspace}(p) = \frac{K_2}{1 + e^{-r_2(p-\mu_2)}},$$

with $K_2 = 100$, $r_2 = 2\times 10^{-2}$, and $\mu_2 = 300\times 10^4$. We also fix the output elasticity with respect to workspace in the production function at 0.05.
We choose the global productivity factor in order to obtain equilibrium wages close to 81.1×10^4 Euro. Therefore, we choose the global productivity factor to be 1.16×10^4 in Test 1 and 0.93×10^4 in Test 2.

Parameter	Test 1	Test 2
d	1	2
α	0.21	0.21
β	0.71	(0.71,0.05)
δ	0.07	0.07
ν	0.1	0.1
ρ	0.1	0.1
A	$1.16.10^4$	$0.93.10^4$
$S_{labour}(w)$	$\frac{6.5}{1+\exp(2.10^{-4}(w-7.10^4))}$	$\frac{6.5}{1+\exp(2.10^{-4}(w-7.10^4))}$
$S_{workspace}(p)$		$\frac{100}{1+\exp(2.10^{-2}(p-300))}$
$\eta(k)$	$\frac{\nu}{\sqrt{2\pi}9.10^4}e^{-\frac{(k-3.10^5)^2}{2(9.10^4)^2}}$	$\frac{\nu}{\sqrt{2\pi}9.10^4}e^{-\frac{(k-3.10^5)^2}{2(9.10^4)^2}}$

Table 2: Summary of parameters used in Test 1 and Test 2.

3.1.2 Numerical results

The table below summarizes the results of the simulations:

	Test 1	Test 2	Data from Insee
Annual production / Nb of firms (in 10^4 Euro)	67.3	67.4	66.4
Total payroll / Nb of firms (in 10^4 Euro)	47.8	47.9	47.1
Nb of Employees / Nb of firms	5.88	5.89	5.81
Annual production / Nb of Employees (in 10^4 Euro)	11.4	11.5	11.4
Total payroll / Nb of Employees (in 10^4 Euro)	8.13	8.13	8.11

Table 3: Comparative table for Test 1 and Test 2.

The equilibrium annual wages for both tests are $w \simeq 8.13 \times 10^4$ Euro. It shows that the employment rate in this sector is

$$\frac{S(w)}{K} \times 100 = 90.6\%.$$

The important outputs of the model are the value function, the distribution the capital of the firms, their optimal consumption, their individual demand on the labour market, and for Test 2 also on the rental market for professionals, and their level of investment. The later figures present these outputs.

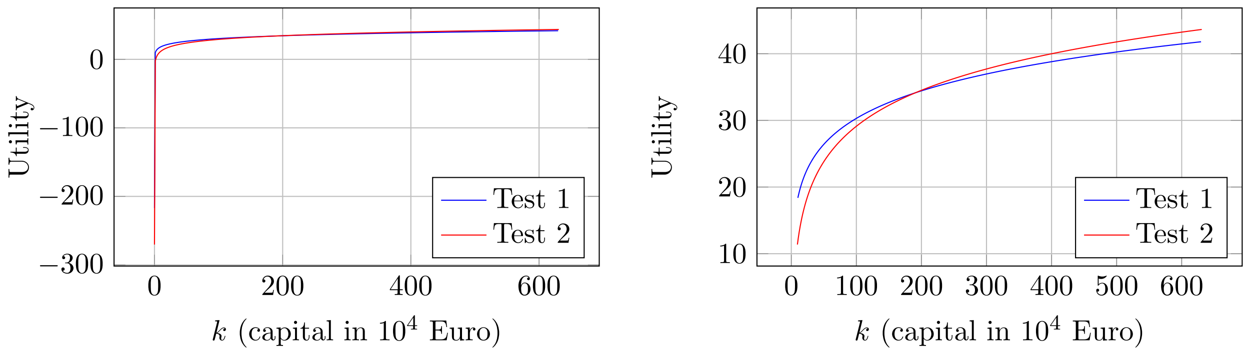


Figure 2: Value functions. This is the value functions corresponding to Test 1 and 2. On the left the curves are displayed on the interval $[10^{-13}, 630]$, and on the right on $[10, 630]$. The value function is strictly increasing and concave. It blows up at $k = 0$: this comes from the choice of the logarithm as a utility function and the fact that the net output vanishes at $k = 0$.

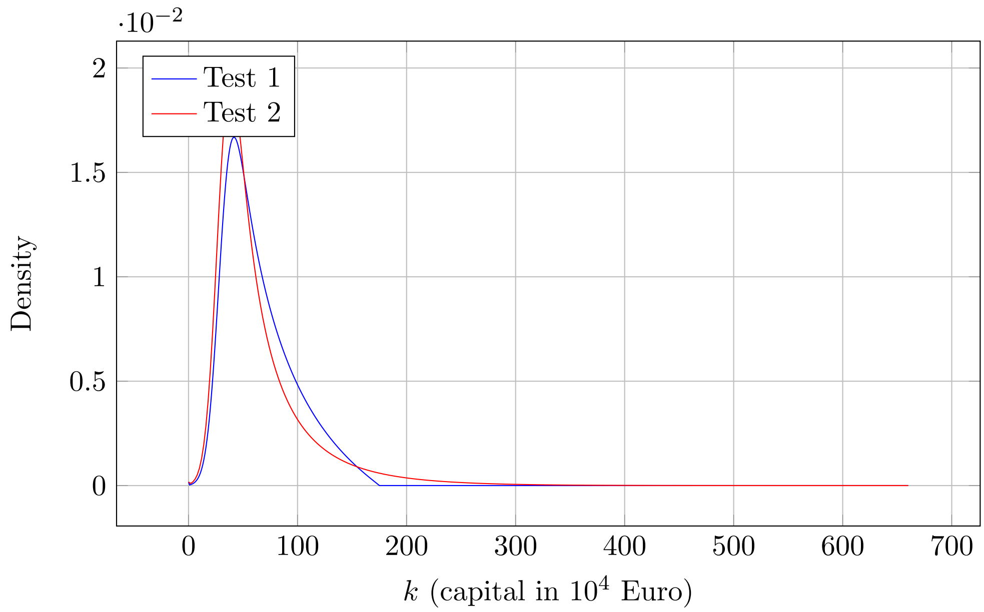


Figure 3: Distributions of capital. The source term η (see Figure 1) explains the peak on the left of the curves. Since the investment is positive on $(0, k^*(w))$ with $k^*(w) \simeq 175 \times 10^4$ Euro for Test 1, and $k^*(w) \simeq 635 \times 10^4$ Euro for Test 2 (see Figure 6), the distribution is shifted to the right. Note that in both cases, the density vanishes at $k = k^*(w)$.

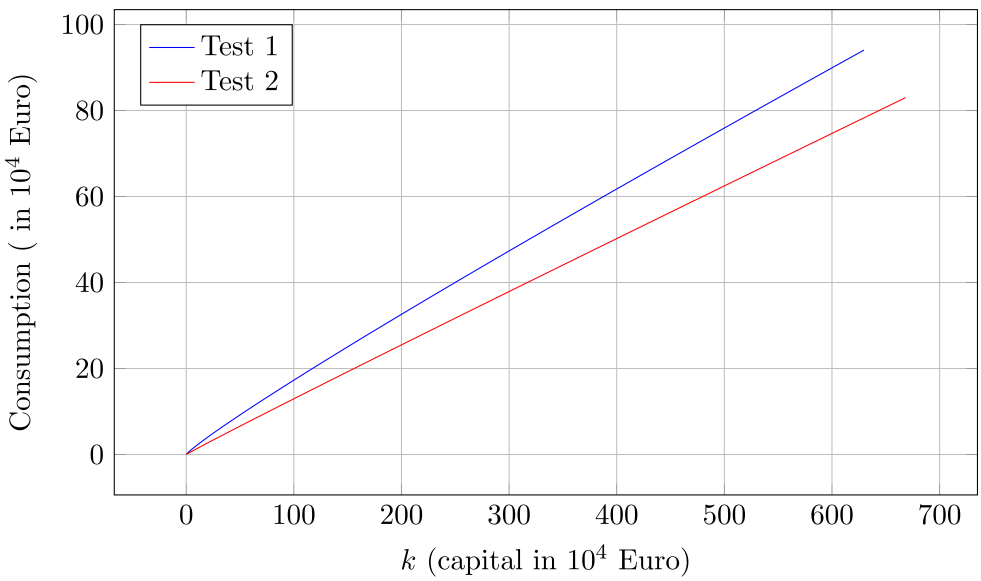


Figure 4: Optimal consumption. The optimal consumption is increasing with respect to the capital. This is linked with the concavity of the value function. Indeed, since for every $k \in (0, +\infty)$, $U'(c(k)) = u'(k)$ where $c(k)$ is the optimal consumption of the firms with capital $k > 0$, then the strict concavity of U and u implies that c is increasing.

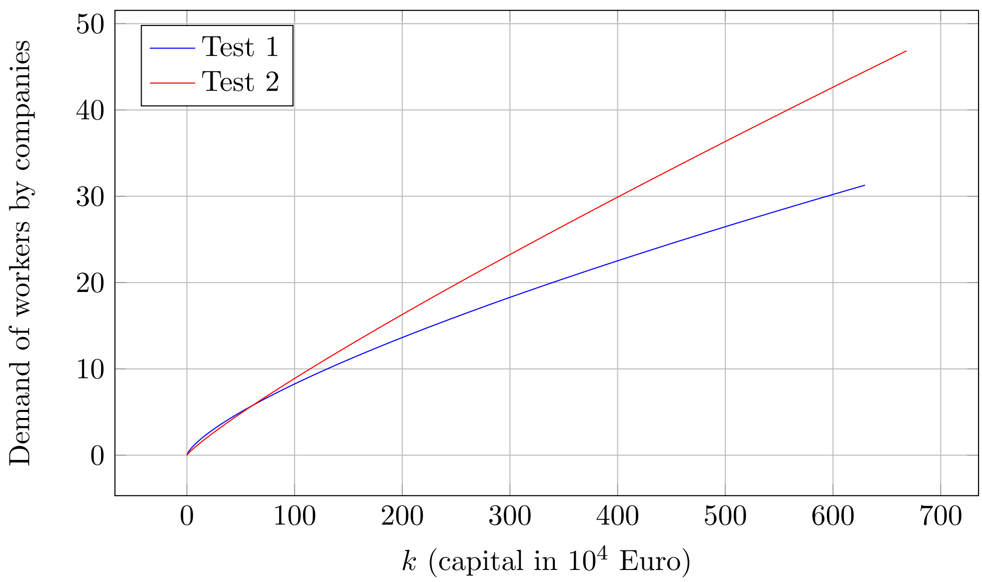


Figure 5: Individual labour demand. The individual labour demand is increasing with respect to the capital.

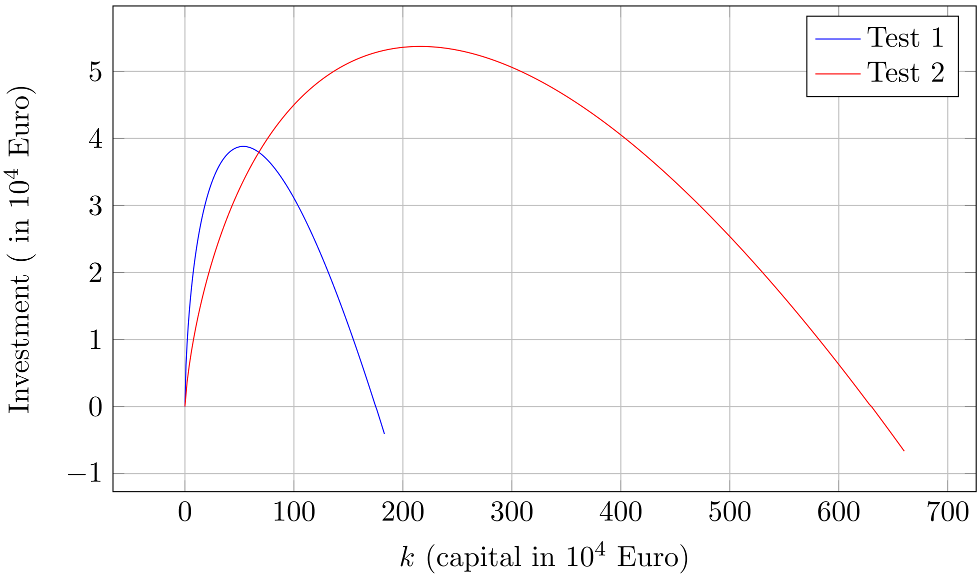


Figure 6: Optimal investment. The curves are smooth on the interval $(0, k^*)$. They also admit a derivative at $k^*(w)$, at least numerically. Since the value of $k^*(w)$ varies much from Test 1 to Test 2, the investment policy differ very much in both tests. However, integrating against m, in average a company in Test 1 accumulates 3.47×10^4 Euro in capital and 3.39×10^4 Euro in Test 2. The relative variation is of 2.31%.

Concerning the rental market for professionals (Test 2), we see that at equilibrium, the rental price per square meter for a year is 391 Euro. We also see that 86.1% of the available workspace is used. Moreover, we observe that the workspace divided by the number of employees is constant with respect to the capital of firms. This is a consequence of the choice of the Cobb-Douglas production function and the optimality conditions which occur in the determination of the net output f. This ratio is equal to 14.6 in this simulation.

From the outputs, it is possible to extract useful data such as the distribution of firms with respect to the number of employees (see the Figure 7 below).

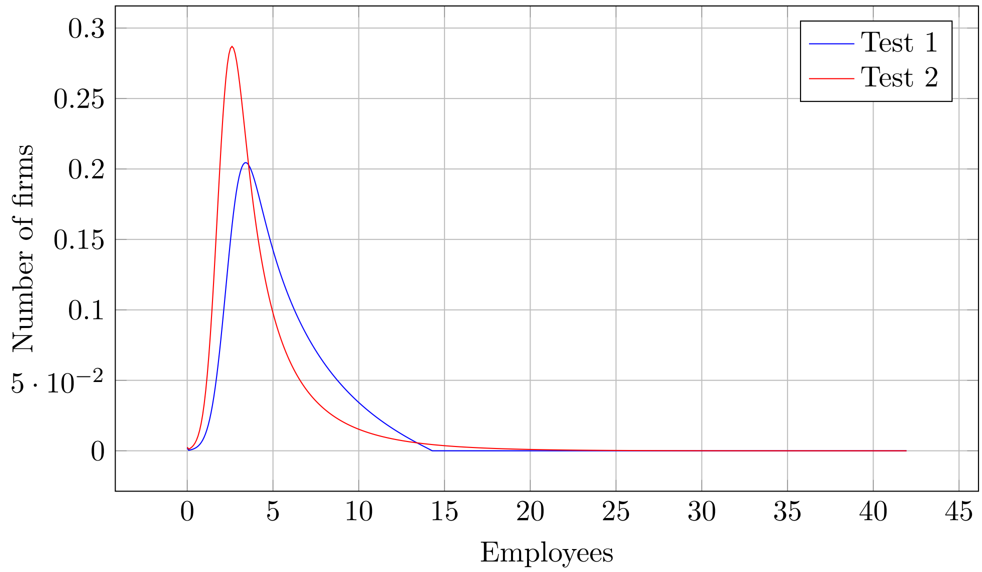


Figure 7: Distribution of firms with respect to the number of employees. We observe that the sizes of the firms in term of number of employees are similar in both tests.

3.1.3 Convergence

Concerning the convergence, we plot the relative residual $\hat{r}_n$, where n refers to the n^{th} iteration, defined by

$$\hat{r}_n = \frac{\Lambda(w^n)}{\Lambda(w^0)}.$$

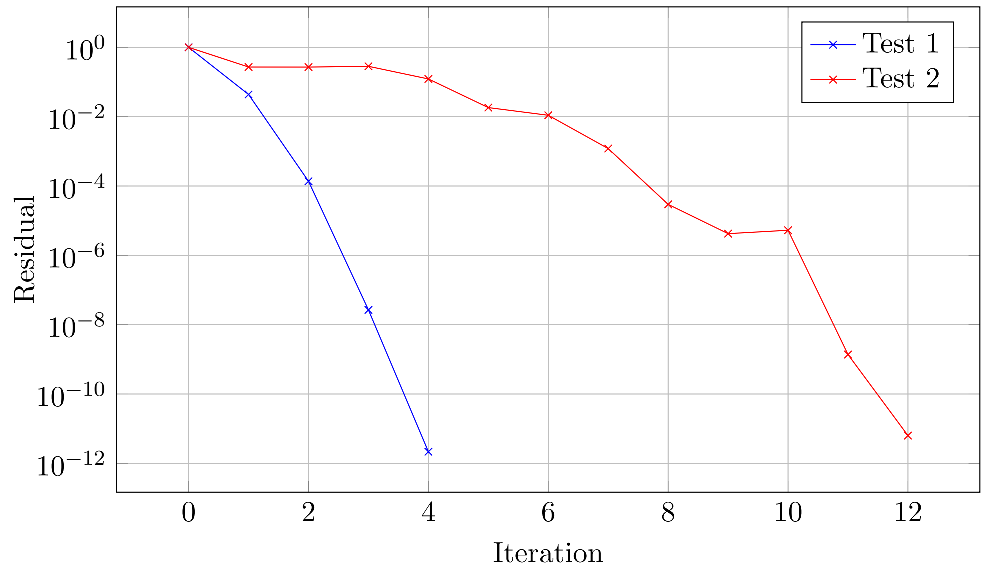
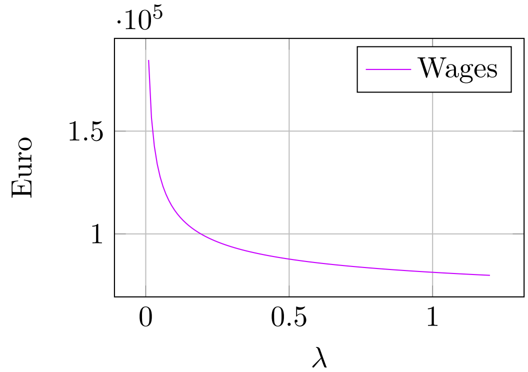


Figure 8: Convergence. To find the equilibrium, we used the secant method in Test 1 and the Broyden's method in Test 2. Note that the convergence is super-linear in both cases and slower in Test 2.

3.2 Sensitivity tests for the combined labour market and the rental market for professionals

We want to understand how the labour market and the rental market for professionals are linked. Note that the labour supply and the workspace supply are independent, i.e. the supply in one market does not depend on the prices on the other market. The curves presented below deal with the sensitivities of the wages and the rental prices. The parameters that we do not modify are the same as those used in Test 2 presented in Table 2.

Sensitivity with respect to the labour supply. As before, we model the labour supply with a function S_{labour} defined in Table 2. Let us introduce a real number λ which will vary between 0.01 to 1.2. We run the test when the labour supply is modelled by $S_\lambda = \lambda S_{labour}$.



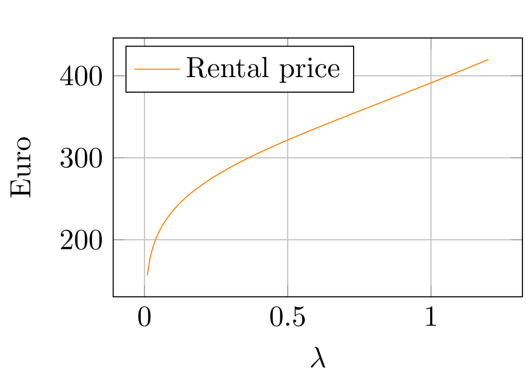
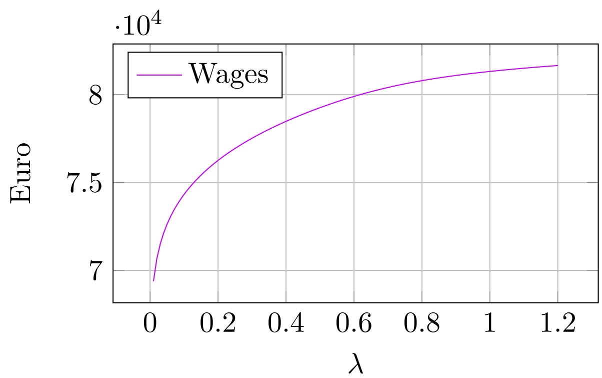


Figure 9: Sensitivity with respect to λ. If λ increases, then the labour supply increases. Thus, the market clearing conditions imply a decrease of the wage. Therefore, firms are more productive which yields an increase of the demand for workspace, which increases the rental price. The wages explode when $\lambda \to 0^+$, while the rental price should vanish. However, we observe that this behaviour is slow since for the value $\lambda = 0.01$ the wages are 184618 Euro and the rental price is 157 Euro.

Sensitivity with respect to the workspace supply. We model the workspace supply with a function $S_{workspace}$ defined in Table 2. Let us introduce a real number λ which will vary between 0.01 to 1.2. We run the test when the workspace supply is modelled by $S_\lambda = \lambda S_{workspace}$.



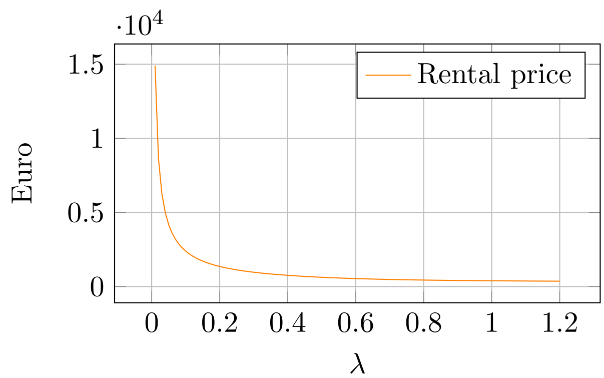


Figure 10: Sensitivity with respect to λ. If λ increases, then the workspace supply increases. Thus, the market clearing conditions imply a decrease of the rental price. Therefore, firms are more productive which yields an increase of the demand for labour, which in turn makes the wages grow. Similarly, in the latter simulation, the rental price explodes when $\lambda \to 0^+$, while the wages should vanish. However, we observe that this behaviour is slow since for the value $\lambda = 0.01$ the rental price is 14913 Euro and the wages are 69390 Euro.

References

[1] Y. Achdou and I. Capuzzo-Dolcetta. Mean field games: numerical methods. *SIAM J. Numer. Anal.*, 48(3):1136–1162, 2010.

[2] Y. Achdou, G. Carlier, Q. Petit, and Tonon D. A mean field model for the interactions between firms on the markets of their inputs. *Math Finan Econ*, 2023.

[3] Conseil Supérieur de l'Audiovisuel. Etude sur le tissu économique du secteur de la production audiovisuelle, 2016.

[4] INSEE. Caractéristiques des services principalement marchands par activité, 2018.

[5] M. Pavol. Sector-specific firm turnover and capital intensity with irreversible investment. 2014.

[6] Q. Petit. *Mean field games and optimal transport in urban modelling*. PhD thesis, PhD thesis, PSL University, 2022.