Journal of Mathematical Economics 111 (2024) 102943

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

An equilibrium model of city with atmospheric pollution dispersion

Mohamed Bahlali [a,*], Quentin Petit [b]

[a] Paris Dauphine University, Climate Economics Chair, Paris, France
[b] EDF R&D, Paris, France

ARTICLE INFO

Manuscript handled by Editor R. Boucekkine

Keywords:
Air pollution
Spatial economics
Urban economics
Atmospheric dispersion

ABSTRACT

We propose a spatial model of city coupling a labour market, a residential market, and air pollution resulting from commuter traffic. The city can be of any shape. Agents choose where to work and live in order to maximize their utility, by consuming goods, residential surface and by valuing air quality. Pollution dispersion is described by an advection–diffusion equation. We prove existence of equilibria, and uniqueness when the number of job locations is finite. We obtain analytical and numerical results emphasizing the combined role of economic and meteorological factors on urban air quality.

1. Introduction

Urban air pollution is a complex and multifaceted problem that requires a comprehensive and interdisciplinary approach to understand and address. Atmospheric dispersion models are powerful tools that are widely used to simulate and predict the dispersion and concentration of pollutants in the air. However, these models alone are not able to capture the human activities that drive pollution emissions. On the other hand, urban economic models can provide valuable insights into the economic activities and land use patterns that contribute to pollution.

The connection between these two complementary approaches is noteworthy, yet still very unexplored. The urban economics theoretical literature that has so far focused on endogenous air pollution (for example, Arnott et al. (2008), Schindler et al. (2017), Regnier and Legras (2018), Kyriakopoulou and Picard (2021)) has largely ignored the phenomenon of atmospheric dispersion. Furthermore, it has relied on stylized spatial settings, often assuming the city is linear and monocentric. According to Wegener (2019), "*Today only few urban models are linked to environmental models to show the impact of planning policies on greenhouse gas emissions, air quality, traffic noise and open space. [...] Even fewer models are able to model the reverse relationship, the impact of environmental quality, such as air quality or traffic noise, on location*".

In this paper, we propose a unifying framework, based on a model developed in Achdou et al. (2023) and, more extensively, in Petit (2022). We consider a closed, plane city of any shape, in which there is a continuum of workers and firms. First, individuals can freely choose where they live and work. They aim at maximizing their utility, by consuming goods, housing surface and by valuing air quality. Second, firms are distributed continuously throughout the city, allowing to model one or several business districts, located anywhere. Third, pollution arises from residential heating, output production and car commuting. Its dispersion is described through an advection–diffusion equation, allowing to account for meteorological effects such as diffusion, transport by wind and lessivage by rain. The source term of this equation depends on where people live and work, and makes the coupling with the housing and labour markets.

As our main result, we prove existence of equilibria, and uniqueness when the number of workplaces is finite. We propose a numerical method for computing solutions.

We then examine the impact of pollution aversion and wind on the equilibrium. Regarding the effect of pollution aversion, we first demonstrate that when the agents are indifferent to pollution, the equilibrium is a Pareto one. Conversely, when they are sensitive to pollution, the equilibrium becomes inefficient, as the agents do not internalize the effects of air pollution into their location decisions. Numerical simulations show that the more residents are sensitive to pollution, the more they tend to concentrate in suburban regions, contributing to increase commuting emissions.

Regarding the role of wind, we show that the level of pollution experienced by residents is determined by both economic and meteorological factors, specifically the relative direction of wind and the revenue gradient. If the wind and revenue gradient are oriented in the same direction, pollution is carried to high-revenue areas, where residents concentrate, resulting in an increase in experienced pollution levels. The reverse conclusion holds if they are oriented in opposite directions.

Our model is quite robust. We make standard assumptions on agents' utility function, commuting cost and firms' demand for labour.

* Corresponding author.
E-mail addresses: mohamed.bahlali@chaireeconomieduclimat.org (M. Bahlali), quentin.petit@edf.fr (Q. Petit).

https://doi.org/10.1016/j.jmateco.2024.102943
Received 6 April 2023; Received in revised form 18 December 2023; Accepted 20 December 2023
Available online 13 January 2024

We account for the main sources of pollution, namely heating, transportation, and output production. Furthermore, our work extends beyond that of Achdou et al. (2023), as we address a continuous distribution of workplaces, in contrast to their model with a discrete number of workplaces. This advance is a step towards an extension of the model to the case where the distribution of companies is endogenous, with firms and residents both competing on the rental property market.

This work relates to a recent strand in the literature that explores the role of spatial externalities resulting from pollution dispersion. Camacho and Pérez-Barahona (2015) examine a problem of optimal consumption and land use in a spatio-temporal setting with pollution diffusion. Focusing on optimal taxation, La Torre et al. (2021) demonstrate that ignoring pollution diffusion can lead to suboptimal policies. Another notable contribution is the work of Boucekkine et al. (2021), who study the problem of optimal productive investment and depollution with transboundary pollution, and extend their findings in a differential game setting in Boucekkine et al. (2022). They use an infinite-dimensional formulation of their optimal control problems to obtain closed-form solutions. Our paper makes three contributions. Firstly, it explores this topic in the field of urban economics, which has not been studied before to the best of our knowledge. Secondly, our model represents a city in two dimensions, while previous models only consider a one-dimensional spatial economy – although their results could be generalized to higher dimensions. Thirdly, our model takes into account advection and diffusion, while previous models (with the exception of Boucekkine et al. (2022)) only consider diffusion in describing the dispersion process.

Some questions remain open. Proving uniqueness in the general case is a difficult exercise and is left for future research. Economically, we ignore the positive externality effect of production (the concentration of employment at a given place increases firms' productivity), which is yet central to the very existence of cities ((Fujita and Ogawa, 1982), Lucas and Rossi-Hansberg (2002)). Furthermore, it would be interesting to expand the model within a dynamic framework; for instance, by incorporating innovation dynamics, accumulation or capital diffusion, or by considering migration frictions. Finally, our description of pollution dispersion can still be improved: for example, we ignore turbulent effects, which have yet an important role in the dispersion process in urban areas, characterized by complex geometries (Bahlali et al., 2019).

As an opening remark, we would like to point out the long-term socioeconomic consequences of pollution advection by wind. For example, it is known that westerly winds in the 19th century contributed to making eastern neighbourhoods of Western European capitals more polluted and deprived than their western counterparts. In some cities, such as Paris or London, this spatial inequality has persisted to our time. Heblich et al. (2021) show that pollution from historical factories in these cities was responsible for 15% of the variation in neighbourhood composition in 1881. Besides, even though these factories have since closed, they show that this past pollution still accounts for up to 20% of the current observed neighbourhood segregation. A possible extension of our model would be to account for heterogeneous agents, in order to capture this spatial inequality. The recent theoretical advances in heterogeneous households models with the use of mean-field games (Achdou et al., 2021) offer an promising avenue for exploration.

The paper is organized as follows. We describe the model in Section 2, and explore a simple case in Section 3. We prove existence of equilibria in Section 4, and uniqueness in Section 5. Section 6 presents some analytical properties of the model. Section 7 is dedicated to the numerical aspects. Section 8 concludes and indicates directions for future research.

Notations. Throughout the paper, the measure of a bounded, Lebesgue-measurable set E of $\mathbb{R}^d$ is denoted by $|E|$, its closure by $\overline{E}$, and its boundary by ∂E. The cardinality of a finite set A is denoted by $\#A$. The notation *a.e.* will stand for *almost everywhere*.

2. Model

We represent the city as an open, convex and bounded subset Ω of $\mathbb{R}^2$, of Lebesgue measure 1, with smooth boundary. There is a continuum of rational resident-workers (or agents). They supply labour and receive wages from competitive firms that produce a unique numéraire good that is both consumed within the city and sold to the larger economy at the same normalized price. Firms are assumed to be immobile and are exogenously concentrated across the city. On the opposite, agents can freely choose where they work and live. They initially select their residential location, followed by their workplace, and finally determine the surface of their accommodation and their consumption level.

2.1. Agents utility and revenue

Consider an individual agent residing at $x \in \overline{\Omega}$ and working at $y \in \overline{\Omega}$. Given a revenue $R(x,y)$, a rental price by surface unit $Q(x)$ and a pollution level $\tilde{E}(x)$ at their residential location, the indirect utility function of this agent is assumed to be

$$U_{\theta,\gamma}\left(R(x,y), Q(x), \tilde{E}(x)\right) = \sup_{C,S} \{C^{\theta} S^{1-\theta} \tilde{E}(x)^{-\gamma},\ C + Q(x)S \leq R(x,y), C \geq 0, S \geq 0\},$$

where $\theta \in [0,1]$ is the preference for consumption, $\gamma \in [0,+\infty)$ is the aversion to pollution exposure, C denotes the level of consumption of the agent, and S the surface of the residence. This utility function is standard and can be found, for example, in Schindler et al. (2017) and Borck (2019). Several empirical studies (Smith and Huang, 1993; Fontenla et al., 2019) have shown that individuals take into consideration the issue of air pollution when making decisions regarding their residential location.

Applying the first-order conditions gives, for any $(R(x,y), Q(x)) \in (0,+\infty)^2$, the optimal consumption and demand for surface, as

$$C_{\theta}(R(x,y)) = \theta R(x,y)$$

and

$$S_{\theta}(R(x,y), Q(x)) = (1-\theta)\frac{R(x,y)}{Q(x)}. \tag{1}$$

For any $(R(x,y), Q(x), \tilde{E}(x)) \in (0,+\infty)^3$, the utility of an agent is therefore given by

$$U_{\theta,\gamma}\left(R(x,y), Q(x), \tilde{E}(x)\right) = \theta^{\theta}(1-\theta)^{1-\theta}\frac{R(x,y)}{Q(x)^{1-\theta}\tilde{E}(x)^{\gamma}}. \tag{2}$$

Agents choose where to work and live in order to maximize their utility, i.e. solve

$$\sup_{x,y\in\overline{\Omega}} U_{\theta,\gamma}\left(R(x,y), Q(x), \tilde{E}(x)\right). \tag{3}$$

According to Eq. (2), the selection of a workplace affects the utility only through the revenue. Thus, an agent will choose the workplace that maximizes his revenue. Given a wage map $w \in C(\overline{\Omega}, \mathbb{R}_+^*)$, agents at the position $x \in \overline{\Omega}$ and working at $y \in \overline{\Omega}$ receive the income $w(y) - c(x,y)$. The map $c \in C(\Omega^2, \mathbb{R}_+)$ represents the commuting cost from x to y.

Assumption 2.1. The function $c : \overline{\Omega}^2 \to \mathbb{R}_+$ is continuous and $c(z,z) = 0$ for every $z \in \overline{\Omega}$.

Therefore, in the absence of any friction, given a wage map $w \in C(\Omega, \mathbb{R}_+^*)$, the revenue of an agent residing at $x \in \overline{\Omega}$ is

$$R(x,w) = \max_{y\in\overline{\Omega}}[w(y) - c(x,y)].$$

In line with spatial economics theoretical literature (Anas, 1990; Alvarez and Lucas, 2007; Allen et al., 2020; Achdou et al., 2023), we incorporate frictions into the model by considering idiosyncratic preferences related to job locations. Let $(\varepsilon(x))_{x\in\overline{\Omega}}$ be a collection of iid standard centred Gumbel random variables. Let us consider:

- $(P^n)_{n\in\mathbb{N}^*}$ a sequence of partitions of $\overline{\Omega}$ such that $\#P^n = n$. P^n can be seen as a partition into n districts of the city $\overline{\Omega}$. We denote by P^n_k the kth element (district) of P^n. It is assumed that every element P^n_k is a connected set.
- for every $n \in \mathbb{N}^*$ and $k \in \{1,\dots,n\}$ p^n_k the Lebesgue measure of P^n_k. We choose $(P^n)_{n\in\mathbb{N}}$ such that $\|p^n\|_\infty \le 2|\overline{\Omega}|/n$.
- for every $n \in \mathbb{N}^*$ and $k \in \{1,\dots,n\}$, y^n_k an element of P^n_k.

Fix $n \in \mathbb{N}^*$. Let us first assume that the agents can choose to work from among the n workplaces $y^n_1,\dots,y^n_k$, and that their preference for location k is given by the random variable $\xi^n(y^n_k) := \varepsilon(y^n_k) - \ln(p^n_k)$.[1] In this case, the expected revenue of an agent residing at $x \in \overline{\Omega}$ writes

$$R^n_\sigma(x,w) = \mathbb{E}\left\{\max_{k=1}^{n}\left[w(y^n_k) - c(x,y^n_k) + \sigma\xi^n(y^n_k)\right]\right\},$$

where $\sigma > 0$ controls the dispersion of idiosyncratic preferences.

We can show (Achdou et al., 2023) that for every $n \in \mathbb{N}^*$,

$$R^n_\sigma(x,w) = \sigma \ln\left(\sum_{k=1}^{n} p^n_k e^{\frac{w(y^n_k)-c(x,y^n_k)}{\sigma}}\right) \tag{4}$$

and, for any $i \in \{1,\dots,n\}$, the probability for the workplace y^n_i to be chosen by an agent living at $x \in \overline{\Omega}$ is

$$\begin{aligned}&\mathbb{P}\left(w(y^n_i) - c(x,y^n_i) + \sigma\xi^n(y^n_i) > w(y^n_k) - c(x,y^n_k) + \sigma\xi^n(y^n_k),\ \forall k \neq i\right)\\ &= \frac{p^n_i e^{\frac{w(y^n_i)-c(x,y^n_i)}{\sigma}}}{\sum_{k=1}^{n} p^n_k e^{\frac{w(y^n_k)-c(x,y^n_k)}{\sigma}}}\end{aligned} \tag{5}$$

Let $Y^n(x)$ be the random variable over $\{y^n_1,\dots,y^n_n\}$ representing the workplace location of an agent residing at $x \in \overline{\Omega}$. The variable $Y^n(x)$ follows the probability distribution (5).

In the continuous case, as the number of workplaces becomes large, we have in the limit

$$\lim_{n\to+\infty} R^n_\sigma(x,w) = \sigma \ln\left(\int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}}dy\right)$$

and, for any continuous bounded function g defined on $\overline{\Omega}$ and taking values in $\mathbb{R}$,

$$\lim_{n\to+\infty} \mathbb{E}\left[g\left(Y^n\right)\right] = \int_\Omega g(y)G_\sigma(x,y,w)dy$$

where

$$G_\sigma(x,y,w) = \frac{e^{\frac{w(y)-c(x,y)}{\sigma}}}{\int_\Omega e^{\frac{w(z)-c(x,z)}{\sigma}}dz}. \tag{6}$$

This means that the sequence $(Y^n(x))_{n\ge1}$ converges in distribution to a continuous random variable with density $G_\sigma(x,\cdot,w)$.

Thus, in the continuous case, for any $x \in \overline{\Omega}$,

$$R_\sigma(x,w) = \sigma \ln\left(\int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}}dy\right) \tag{7}$$

can be seen as the expected revenue of an agent residing at x, and $G_\sigma(x,y,w)$ as the probability density for an agent located at x to choose the workplace $y \in \overline{\Omega}$. In Appendix, we show that $R_\sigma(x,w)$ converges to $R(x,w)$ when σ goes to 0.

In the end, an individual agent chooses his residential location in order to solve

$$\sup_{x\in\overline{\Omega}} U_{\theta,\gamma}\left(R_\sigma(x,w), Q(x), \tilde{E}(x)\right). \tag{8}$$

[1] The correcting term $-\ln(p^n_k)$ ensures that in the absence of income, $\mathbb{E}\left\{\max_{k=1}^{n}\left[\xi^n(y^n_k)\right]\right\} = 0$.

2.2. Labour market

Let $\mathcal{P}_c(\Omega)$ be the set of probability measures on $\overline{\Omega}$ that admit a continuous density with respect to the Lebesgue measure. For any distribution of residents $\mu \in \mathcal{P}_c(\Omega)$ and any wage function $w \in C(\Omega,\mathbb{R}^*_+)$, the density of labour supply in $y \in \overline{\Omega}$ is given by

$$\int_\Omega G_\sigma(x,y,w)d\mu(x).$$

By the law of total probability, this is simply the integral, over all the living places $x \in \overline{\Omega}$, of the density of residents at x, multiplied by the probability density for an agent to work at y knowing that they reside at x.

On the demand side, let $F : \mathbb{R}_+ \to \mathbb{R}$ be a production function satisfying the usual Inada conditions.

Assumption 2.2 (*Inada Conditions*)**.** The function $F : \mathbb{R}_+ \to \mathbb{R}$ satisfies $F(0) = 0$, F is strictly concave on $\mathbb{R}_+$, $\lim_{x\to\infty} F'(x) = 0$ and $\lim_{x\to 0} F'(x) = +\infty$.

Define the profit of a firm as

$$\pi(v) := \sup_{l\ge0}\left\{F(l) - lv\right\}, \tag{9}$$

where l represents the quantity of labour and v the wage.

The labour demand of an individual firm is

$$\ell(v) = F'^{-1}(v) = -\pi'(v).$$

We see that ℓ is differentiable, decreasing and such that $\lim_{v\to0^+} \ell(v) = +\infty$ and $\lim_{v\to+\infty} \ell(v) = 0$.

Let $\nu : \overline{\Omega} \to \mathbb{R}$ be the spatial concentration of firms.

Then, the aggregate labour demand at a certain location $y \in \overline{\Omega}$, where the wage is $v > 0$, is given by

$$L(y,v) = \nu(y)\ell(v).$$

The labour market clearing condition thus writes as follows:

$$\int_\Omega G_\sigma(x,y,w)d\mu(x) = L(y,w(y)), \quad \forall y \in \overline{\Omega}. \tag{10}$$

2.3. Housing market

Now, for any distribution of residents $\mu \in \mathcal{P}_c(\Omega)$, any wage function $w \in C(\Omega,\mathbb{R}^*_+)$ and any rental price function $Q \in C(\Omega,\mathbb{R}_+)$, the aggregate demand for surface is given, for all $x \in \overline{\Omega}$, by

$$S_\theta(R_\sigma(x,w), Q(x))\mu(x).$$

It is the individual demand for surface, given by (1), multiplied by the density of residents at x.

We assume that the housing supply is exogenous and given by a function ψ satisfying the following assumption.[2]

Assumption 2.3. The housing supply $\psi : \overline{\Omega} \to \mathbb{R}_+$ is continuous and takes positive values.

The housing market clearing condition then writes as follows:

$$S_\theta(R_\sigma(x,w), Q(x))\mu(x) = \psi(x), \quad \forall x \in \overline{\Omega}. \tag{11}$$

[2] The results of this paper still hold if we consider an isoelastic supply $\phi(Q) = Q^\rho$ with $\rho > 0$.

2.4. Pollution dispersion

We assume that the pollution concentration $\tilde{E}(z)$ at $z \in \overline{\Omega}$ can be decomposed into two terms: a background regional level $E_0 > 0$ and a local level $E(z)$, such that $\tilde{E}(z) = E_0 + E(z)$.[3]

The dispersion of local pollution involves several physical and chemical processes, the main ones being:

- *Advection*, which refers to the transport of pollution by wind;
- *Molecular diffusion*, which reflects that pollution naturally spreads from high concentration to low concentration areas;
- *Chemical interactions* between the emitted pollutants and chemical species in the air;
- *Lessivage*, which is the process of natural air purification (for example, by rain).

If we neglect the chemical interactions, the stationary distribution of E solves the following scalar transport equation (Sportisse, 2009):

$$\underbrace{\mathbf{V}(z)\cdot\nabla E(z)}_{advection} = \underbrace{\nabla\cdot(k\nabla E(z))}_{diffusion} + \underbrace{f(z)}_{source\ term} - \underbrace{\lambda E(z)}_{lessivage}, \quad \forall z \in \Omega, \tag{12}$$

where $\mathbf{V}(z) \in \mathbb{R}^2$ is the wind field at $z \in \Omega$, $k \in (0,+\infty)$ the diffusion coefficient, and $\lambda \in (0,+\infty)$ the lessivage coefficient. The wind field $\mathbf{V}$ satisfies an incompressibility condition: for all $z \in \Omega$, $\nabla\cdot\mathbf{V}(z) = 0$. We consider an elliptic equation rather than a time-dependent one[4] because the characteristic timescale of pollution dispersion is much shorter compared to the characteristic timescale of economic decisions. Typically, the timescales involved in urban-scale pollutant dispersion range from several hours to a few days at most (Lucas, 1958), while the timescales associated with economic decisions (such as choosing a residence or workplace) are typically on the order of years.

We assume, without loss of generality, that $k = 1$. The dispersion equation then becomes:

$$\Delta E(z) - \mathbf{V}(z)\cdot\nabla E(z) - \lambda E(z) + f(z) = 0.$$

The only thing left is to clarify the source term $f(z)$. We consider three sources of emissions: residential heating, output production and car commuting. The pollution resulting from residential heating is assumed to be proportional to the housing supply $\psi(z)$. The pollution resulting from the production of the output is assumed to be proportional to $F(\ell(w(z)))$. Finally, we assume that the commuting path is a straight line from home to work. The road network is very dense and can be viewed as a continuum. It has a certain width $\delta > 0$. For any $z \in \Omega$, we denote by $\phi_{\mu,w}$ the flux of individuals commuting by the element of road at z. This is the sum of all the agents passing through z. In other words, it is the sum over all workplaces and residential locations of the probability that agents will cross z by following their path to work, i.e.

$$\phi_{\mu,w}(z) = \int_{\overline{\Omega}^2} \delta^{-1}\mathbf{1}_{z\in\Sigma_\delta(x,y)}\mu(x)G_\sigma(x,y,w)\,dx\,dy, \tag{13}$$

where $\Sigma_\delta(x,y) := \{s \in \Omega,\ \exists t \in [x;y],\ |t-s| \le \delta\}$ is the surface of the rectangle of length $|x-y|$, of width δ, centred around the segment $[x;y]$. The source term thus writes, for all $z \in \Omega$

$$f_{\mu,w}(z) = \alpha_1\psi(z) + \alpha_2\nu(z)F(\ell(w(z))) + \alpha_3\phi_{\mu,w}(z), \tag{14}$$

where $(\alpha_1,\alpha_2,\alpha_3) \in [0,+\infty)^3$. The previous equation then writes:

$$\Delta E(z) - \mathbf{V}(z)\cdot\nabla E(z) - \lambda E(z) + f_{\mu,w}(z) = 0.$$

The value of local pollution at the boundary is supposed to be zero: the borders of the city correspond to rural areas with little pollution. Thus, for all $s \in \partial\Omega$, $E(s) = 0$.

The equation for local pollution dispersion finally takes the following form

$$\begin{cases} \Delta E(z) - \mathbf{V}(z)\cdot\nabla E(z) - \lambda E(z) + f_{\mu,w}(z) = 0, & \forall z \in \Omega, \\ E(s) = 0, & \forall s \in \partial\Omega. \end{cases} \tag{15}$$

We will consider weak solutions to Eq. (15), as defined in the following. We denote by $H_0^1(\Omega)$ the first order Sobolev space on Ω with zero boundary value.

Definition 2.1. We say that $u \in H_0^1(\Omega)$ is a weak solution to (15) if for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u\cdot\nabla v + \int_\Omega (\mathbf{V}\cdot\nabla u)\,v + \lambda\int_\Omega uv = \int_\Omega f_{\mu,w}v.$$

2.5. Equilibrium

We define an equilibrium as follows.

Definition 2.2. We say that $(w,Q,E,\mu) \in C(\Omega,\mathbb{R}_+^*)^2 \times \left(H_0^1(\Omega) \cap C(\Omega,\mathbb{R}_+^*)\right) \times \mathcal{P}_c(\Omega)$ is an equilibrium if

$$\int_\Omega G_\sigma(x,y,w)d\mu(x) = L(y,w(y)), \quad \forall y \in \overline{\Omega}, \tag{16}$$

$$S_\theta(R_\sigma(x,w),Q(x))\mu(x) = \psi(x), \qquad \forall x \in \overline{\Omega}, \tag{17}$$

$$-\Delta E(z) + \mathbf{V}(z)\cdot\nabla E(z) + \lambda E(z) = f_{\mu,w}(z), \qquad \forall z \in \Omega, \tag{18}$$

$$\operatorname{supp}\mu \subset \operatorname*{argmax}_{x\in\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w),Q(x),E(x)), \tag{19}$$

where (18) is completed with the Dirichlet condition $E = 0$ on $\partial\Omega$.

In Definition 2.2, Eq. (16) reflects the equilibrium in the labour market, (17) the one in the housing market, (18) the dispersion of pollution, and (19) is a mobility condition: residents choose to locate at places that maximize their utility. This condition implies that at the equilibrium, all the agents get the same utility level. It can also be seen as a Nash equilibrium condition. Indeed, if $\overline{\Omega} \ni x \mapsto U_{\theta,\gamma}(R_\sigma(x,w),Q(x),\tilde{E}(x))$ is continuous, then (19) is equivalent to

$$\int_\Omega U_{\theta,\gamma}(R_\sigma(x,w),Q(x),\tilde{E}(x))d\mu(x) = \sup_{\nu\in\mathcal{P}_c(\Omega)} \int_\Omega U_{\theta,\gamma}(R_\sigma(x,w),Q(x),\tilde{E}(x))d\nu(x), \tag{20}$$

which is a mean-field equation characterizing a Nash equilibrium with a continuum of players.

3. An explicit solution in a simple case

We first apply our model to a simple case: the linear monocentric city. We neglect the effects of diffusion and wind. These approximations allow us to obtain, in this one dimensional case, an explicit formulation of the equilibrium.

Let us consider the segment $[0,1]$ as our linear city. There is only one working place, located in 1. The wage function then reduces to one single value, w^*, which is solution to a simple labour equation

$$L(w^*) = 1 \tag{21}$$

The revenue of an agent located in $x \in [0,1]$ is $R(x,w^*) = w^* - c(x)$. The housing equation writes, for all $x \in [0,1]$

$$(1-\theta)R(x,w^*)\mu(x) = Q(x) \tag{22}$$

[3] Background pollution originates at a larger scale and is independent from local emissions (Tchepel et al., 2010).

[4] In a temporal setting, the evolution equation satisfied by $E(z,t)$ would write:

$$\partial_t E(z,t) + \mathbf{V}(z,t)\cdot\nabla E(z,t) = \nabla\cdot(k\nabla E(z,t)) + \chi(E(z,t);z) + f(z,t) - \lambda E(z,t), \quad \forall z \in \Omega.$$

.

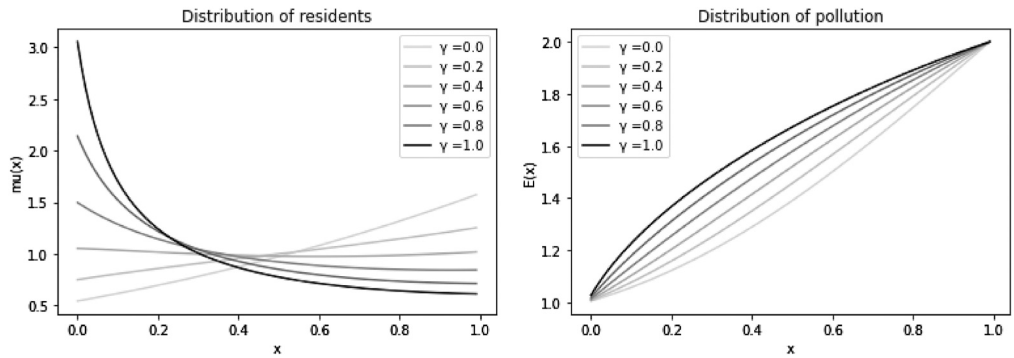

Fig. 1. Equilibrium in the 1D monocentric city, for different values of γ.

Because we ignore diffusion and advection, we have, for all $x \in [0,1]$

$$\lambda \tilde{E}(x) = \lambda E_0 + f_\mu(x).$$

The source term $f_\mu(x)$ is the density of people commuting by x. As all the agents work in 1, we have

$$f_\mu(x) = \int_0^x \mu(s)\,ds$$

and the pollution equation becomes

$$\tilde{E}(x) = E_0 + \lambda^{-1} \int_0^x \mu(s)\,ds \tag{23}$$

Finally, the mobility condition writes

$$\operatorname{supp} \mu \subset \operatorname*{argmax}_{x\in[0,1]} U_{\theta,\gamma}(R_\sigma(x,w), Q(x), E(x)) \tag{24}$$

Any solution $(w^*, Q, E, \mu) \in \mathbb{R}_+^* \times C([0,1],\mathbb{R}_+^*) \times C^1([0,1],\mathbb{R}_+^*) \times \mathcal{P}_c([0,1])$ to the system given by (21), (22), (23) and (24) is an equilibrium of our problem.

The following Proposition gives existence and uniqueness of the equilibrium. In addition, it says that pollution increases with agents' aversion to pollution.

Proposition 3.1. *Assume that the function c is continuous, decreasing, $c(1) = 0$ and $c(0) < L^{-1}(1)$. The system formed by (21), (22), (23) and (24) admits a unique equilibrium, where the pollution is explicitly given by*

$$\tilde{E}(x) = \left[E_0^{\frac{1+\gamma-\theta}{1-\theta}} + \left(\left(E_0 + \lambda^{-1}\right)^{\frac{1+\gamma-\theta}{1-\theta}} - E_0^{\frac{1+\gamma-\theta}{1-\theta}} \right) \frac{\int_0^x R(s,w^*)^{\frac{\theta}{1-\theta}}\,ds}{\int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}}\,ds} \right]^{\frac{1-\theta}{1+\gamma-\theta}},$$
$$\forall x \in [0,1] \tag{25}$$

There exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$, $d\tilde{E}(x)/d\gamma > 0$ for all $x \in [0,1]$. In other words, at the equilibrium, pollution increases with the aversion of the population to pollution, γ.

The intuition behind Proposition 3.1 is simple. The more individuals are pollution averse, the more they tend to move away from it by living far from the city centre. But, in doing so, they increase their commuting distance, thus the amount of pollution they release.

To illustrate this effect, we compute the equilibrium in pollution, residence, and rental price, for several values of γ, ranging from 0 to 1. We choose L such that $w^* = L^{-1}(1) = 1$. We assume a linear commuting cost function, i.e. $c(x) = c_0(1-x)$, with $c_0 = 0.3$, and set other parameters as follows: $\theta = 0.75$, $\lambda = 1.0$, $E_0 = 1.0$. Fig. 1 displays the numerical results. As γ increases, agents tends to concentrate in 0, away from the city centre in 1, raising in turn the total amount of pollutants released.

We already observe, through this simple example, that the parameter γ is an important factor in the model. In Section 6.1, we demonstrate that when it is equal to zero, the resulting equilibrium is a Pareto one. In Section 7.4.1, we provide additional numerical examples illustrating that as γ increases, it can motivate agents to reside far from polluted areas, resulting in longer commuting distances. Thus, the parameter γ has a dual impact on utility: (1) by making agents sensitive to the pollution externality and (2) by influencing residential location choices, consequently affecting pollution at the equilibrium.

4. Existence in the general case

In this section, we aim to prove existence of equilibria in the general case, as described in Section 2. We assume that Assumption 2.1, 2.2 and 2.3 are in force. We assume furthermore that the transportation cost c is of class C^1 on Ω^2, and that the spatial concentration of firms ν is differentiable, and satisfies $1/\eta \le \nu \le \eta$, for some positive η.

The following theorem is the main result of the paper.

Theorem 4.1. *There exists at least one equilibrium, in the sense of Definition 2.2.*

The proof, presented in the subsequent part of this section, is inspired by Achdou et al. (2023). It relies on a fixed-point argument: we build a continuous map $\mathcal{Y}$ such that the fixed-points of $\mathcal{Y}$ are exactly the solutions of the equilibrium problem. To that end, we first show that the distribution of residents, μ, can be explicitly obtained from the wage function w and the pollution $\tilde{E}$. We then show that the solutions w and $\tilde{E}$ belong to convex and compact subsets of, respectively, $C(\overline{\Omega},\mathbb{R})$ and $L^2(\Omega)$. We then apply Schauder fixed point theorem:

Theorem (*Schauder Fixed Point Theorem*)**.** *Let F be a normed vector space, K a convex and compact subset of F and $\mathcal{Y}$ a continuous application from K into itself. Then $\mathcal{Y}$ admits at least one fixed-point.*

We start by showing that any equilibrium wage map belongs to a convex and compact subset of $C(\Omega,\mathbb{R})$. To that end, following Petit (2022), we show that any solution to (16) can be expressed as the unique solution to a convex minimization problem.

Proposition 4.1. *For any distribution of agents $\mu \in \mathcal{P}_c(\Omega)$, (16) holds for $w \in C(\Omega,\mathbb{R}_+^*)$ if and only if w is the unique minimizer of*

$$\min_{z\in K_1} \left\{ \int_\Omega R_\sigma(x,z)\,d\mu(x) + \int_\Omega \nu(y)\pi(z(y))\,dy \right\}. \tag{26}$$

where

$$K_1 := \left\{ z \in C^1(\Omega,\mathbb{R}_+),\ z(\cdot) \ge \ell^{-1}(\eta),\ \|z\|_{L^\infty} \le M_1,\ \|\nabla z\|_{L^\infty} \le M_2 \right\}$$

and M_1 and M_2 are independent of μ. Besides, K_1 is convex and compact in $C(\Omega,\mathbb{R})$.

The outline of the proof is as follows. We first provide a priori bounds on the solutions to (26) and their derivatives to reduce the minimization problem to a compact and convex subset of $C(\Omega,\mathbb{R})$. We then apply the direct method in the calculus of variations to deduce the existence of a unique solution.

Remark. The variational problem (26) takes a labour demand perspective, as $\ell(w(y))$ represents the individual labour demand of a firm at the equilibrium wage $w(y)$. The following Proposition formulates an equivalent variational problem from a labour supply perspective, which is essentially the dual of the first problem. The proof is given in Appendix.

Proposition 4.2. *For any $\mu \in \mathcal{P}_c(\Omega)$, if $w \in C(\Omega, \mathbb{R}_+^*)$ solves (16), then w solves*

$$\max_{w \in K_1} \left\{ \int_\Omega R_\sigma(x, w) d\mu(x) + \int_\Omega \nu(y) \Pi_{\mu,w}(y) dy \right\} \tag{27}$$

where, for all $y \in \overline{\Omega}$ and $w \in C(\Omega, \mathbb{R}_+^*)$,

$$\Pi_{\mu,w}(y) := F\left(\nu(y)^{-1} l_{\mu,w}(y)\right) - l_{\mu,w}(y) w(y),$$

and

$$l_{\mu,w}(y) := \int_\Omega G_\sigma(x, y, w) d\mu(x).$$

Thus, given the commuting probabilities $G_\sigma(x, y, w)$, the clearing wage maximizes the total surplus of firms and workers. This result is economically not surprising, because the labour market is not affected by the pollution externality and we assumed perfect competition between firms.

Now, as usual in quantitative urban models (Diamond, 2016), we make use of the housing market clearing condition and free mobility of the agents to obtain an explicit formulation of the equilibrium distribution of residents.

Lemma 4.1. *Let $(w, Q, E, \mu) \in C(\Omega, \mathbb{R}_+^*)^2 \times H_0^1(\Omega) \cap C(\Omega, \mathbb{R}_+^*) \times \mathcal{P}_c(\Omega)$ be an equilibrium. Then,*

$$\mu(x) = \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}} \psi(x)}{\int_\Omega R_\sigma(y, w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{-\frac{\gamma}{1-\theta}} \psi(y)\, dy}, \quad \forall x \in \overline{\Omega}. \tag{28}$$

Eq. (28) displays an equilibrium relationship between the distributions of residents, wages and pollution: people tend to locate where revenues are high and pollution is low. As a consequence, if (w, Q, E, μ) is an equilibrium, then the source term of the pollution equation, $f_{\mu,w}$, can be expressed as a function depending on w and E, i.e. for all $z \in \overline{\Omega}$

$$\begin{aligned} f_{\mu,w}(z) = f_{w,E}(z) &:= \alpha_1 \psi(z) \\ &+ \alpha_2 \nu(z) F(\ell(w(z))) \\ &+ \alpha_3 \int_{\Omega^2} \delta^{-1} \mathbf{1}_{z \in \Sigma(x,y)} \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_\Omega R_\sigma(s, w)^{\frac{\theta}{1-\theta}} \tilde{E}(s)^{-\frac{\gamma}{1-\theta}}\, ds} G_\sigma(x, y, w) dx dy \end{aligned}$$

Given this new formulation of the pollution source term, we finally show that any equilibrium distribution of pollution belongs to a convex and compact subset of $L^2(\Omega)$.

Proposition 4.3. *Let $(w, E) \in C(\Omega, \mathbb{R}_+^*) \times H_0^1(\Omega) \cap C(\Omega, \mathbb{R}_+^*)$. The PDE*

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = f_{w,E}(z), & \forall z \in \Omega, \\ u(s) = 0, \quad \forall s \in \partial\Omega, \end{cases} \tag{29}$$

admits a unique solution $u_{w,E} \in H_0^1(\Omega)$. Moreover, $u_{w,E}$ is positive and belongs to

$$K_2 = \left\{ u \in H_0^1(\Omega),\ \|\nabla u\|_{L^2} \le |\Omega|\, \delta^{-2} \min(1, \lambda)^{-1} \right\},$$

which is convex and compact in $L^2(\Omega)$.

The proof relies on applying Riesz's representation theorem for the existence and uniqueness part, Hölder inequality for the majoration of the solution derivative, and Rellich's theorem for the compactness of K_2.

We shall now use a fixed-point argument to establish the existence of an equilibrium. Proposition 4.4 builds a map $\mathcal{Y}$ defined on $K_1 \times K_2$ whose fixed-points are exactly the equilibria.

Proposition 4.4. *Let us define the function $\mathcal{Y} : K_1 \times K_2 \to K_1 \times K_2$ by the following construction:*

(1) To any $(w, E) \in K_1 \times K_2$, we associate the probability $\mu(w, E)$ on Ω with density

$$\Omega \ni x \mapsto \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}} \psi(x)}{\int_\Omega R_\sigma(y, w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{-\frac{\gamma}{1-\theta}} \psi(y)\, dy}, \tag{30}$$

with respect to the Lebesgue measure.

(2) We define $\mathcal{Y}_1(w, E)$ as the unique solution to (26) associated to $\mu(w, E)$, i.e. $\mathcal{Y}_1(w, E)$ is the unique minimizer of

$$\min_{z \in \mathring{K}_0} \left\{ \phi_{\mu(w,E)}(z) - \int_\Omega \int_\epsilon^{z(y)} L(y, s) ds dy \right\}.$$

(3) We define $\mathcal{Y}_2(w, E)$ as the unique solution to (29), i.e.

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z) \cdot \nabla u(z) + \lambda u(z) = f_{w,E}(z), & \forall z \in \Omega, \\ u(s) = 0, & \forall s \in \partial\Omega. \end{cases}$$

The fixed points of $\mathcal{Y}$ are exactly the equilibria, in the sense of Definition 2.2.

Proof. First, $\mathcal{Y}$ is well defined because the solutions to (26) and (29) respectively belong to K_1 and K_2. Now, if $(w, E) = \mathcal{Y}(w, E)$, let us consider μ the probability measure given by (30), and the rental price $Q : \overline{\Omega} \to \mathbb{R}_+^*$ given by

$$Q(x) = (1 - \theta) R_\sigma(x, w) \mu(x), \quad \forall x \in \Omega$$

The quadruplet $(w, Q, E, \mu) \in K_1 \times C(\Omega, \mathbb{R}_+^*) \times K_2 \times \mathcal{P}_c(\Omega)$ is an equilibrium since (16) holds because of $w = \mathcal{Y}_1(w, E)$ and Proposition 4.1, (17) holds by definition of Q, (18) holds because $E = \mathcal{Y}_2(w, E)$, and for all $x \in \Omega$,

$$U_{\theta,\gamma}(R_\sigma(x, w), Q(x), E(x)) = \theta^\theta \left(\int_\Omega R_\sigma(z, w)^{\frac{\theta}{1-\theta}} \tilde{E}(z)^{\frac{-\gamma}{1-\theta}} \psi(z) dz \right)^{1-\theta}$$

a constant value which implies that the mobility condition (19) holds. Finally, if (w, Q, E, μ) is an equilibrium, from Lemma 4.1 μ is given by (30), w is the solution to (26) associated to μ, and E is the unique solution to (29) associated to (w, E). Therefore $(w, E) = \mathcal{Y}(w, E)$. □

The mapping $\mathcal{Y}$ takes only two arguments: w and E. These two variables are enough to characterize an equilibrium, because Eqs. (11) and (28) relate them with Q and μ. Proposition 4.5 establishes continuity of $\mathcal{Y}$.

Proposition 4.5. *The map $\mathcal{Y}$ is continuous on $(K_1, \|\cdot\|_{L^\infty}) \times (K_2, \|\cdot\|_{L^2})$.*

The outline of the proof is as follows. To establish the continuity of $\mathcal{Y}_1$, we first prove the continuity of the equilibrium distribution of residents, explicitly given by Lemma 4.1, with respect to w and E. Then, we prove the continuity of the solutions to problem (26) with respect to μ. To establish the continuity of $\mathcal{Y}_2$, we first prove the continuity of the source term $f_{w,E}$ with respect to w and E, and then prove the continuity of the solutions to the scalar transport Eq. (29) with respect to the source term.

We are now able to prove our main theorem. We recall it here:

Theorem. *There exists at least one equilibrium, in the sense of Definition 2.2.*

Proof. By Proposition 4.5, the map $\mathcal{Y}$ is continuous from the convex and compact set $K_1 \times K_2$ into itself. By Schauder's fixed-point theorem, $\mathcal{Y}$ admits at least one fixed-point. Therefore, by Proposition 4.4, there exists at least one equilibrium, in the sense of Definition 2.2. □

5. Existence and uniqueness in a semi-discrete case

We consider in this section a semi-discrete version of the model, where the distribution of firms ν is the sum of a finite number of Dirac masses. In other words, there is a finite number of job locations, as in Achdou et al. (2023). This more specific framework allows to obtain additional results: we can prove both existence and uniqueness of equilibria, and we also study the case where there are no idiosyncratic preferences ($\sigma = 0$).

Let us present the assumptions and notations related to this setting. We assume that there are $N \in \mathbb{N}^*$ workplaces. The commuting cost is now a set of functions $c_i : \overline{\Omega} \to \mathbb{R}_+$, for $i \in \{1, \dots, N\}$, where $c_i(x)$ corresponds to the transportation costs to reach the ith workplace coming from $x \in \overline{\Omega}$.

Assumption 5.1. For every $i \in \{1, \dots, N\}$, the commuting cost associated to the ith workplace, c_i, is continuous.

The wages and spatial concentration of firms become two N-uplets $(w_1, \dots, w_N)$ and $(\nu_1, \dots, \nu_N)$ belonging to $(0, +\infty)^N$, such that the demand for labour at the ith workplace where the wage is w_i now writes $L_i(w_i) = \nu_i \ell(w_i)$.

According to formula (4), given a collection of wages $w \in (0, +\infty)^N$, the expected revenue of an agent living at $x \in \overline{\Omega}$ now writes

$$R_\sigma^N(x, w) = \sigma \ln \left(\frac{1}{N} \sum_{i=0}^{N} e^{\frac{w_i - c_i(x)}{\sigma}} \right),$$

where $\sigma > 0$ controls the dispersion of idiosyncratic preferences.[5] We assume that the workplace number 0 ensures, for each agent, a fixed salary $\overline{w}_0$ and a positive revenue, i.e. $\overline{w}_0 - c_0(x) > 0$ for all $x \in \overline{\Omega}$.

According to (5), the probability for an agent at the position $x \in \overline{\Omega}$ to choose the workplace $i \in \{0, \dots, N\}$ is now given by the discrete Gibbs distribution:

$$G_{\sigma,i}(x, w) = \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{j=0}^{N} e^{\frac{w_j - c_j(x)}{\sigma}}}.$$

The pollution source term $f_{\mu,w}$ is now, for all $z \in \Omega$:

$$\begin{aligned} f_{\mu,w}(z) = {} & \alpha_1 \psi(z) + \alpha_2 \sum_{i=1}^{N} \mathbf{1}_{z = y_i} \nu_i F(\ell(w_i)) \\ & + \alpha_3 \sum_{i=1}^{N} \int_\Omega \delta^{-1} \mathbf{1}_{z \in \Sigma_{i,\delta}(x)} \mu(x) G_{\sigma,i}(x, w) dx \end{aligned} \tag{31}$$

where $y_i \in \overline{\Omega}$ corresponds to the coordinate of the ith workplace, and $\Sigma_{i,\delta}(x) := \{s \in \Omega,\ \exists t \in [x; y_i],\ |t - s| \le \delta\}$ is the surface of the rectangle of length $|x - y_i|$, of width δ, centred around the segment $[x; y_i]$.

Definition 5.1 reformulates the equilibrium problem in this semi-discrete case.

Definition 5.1. When the number of workplaces is finite, we say that $(w, Q, E, \mu) \in (0, +\infty)^N \times C(\Omega, \mathbb{R}_+^*) \times \left(H_0^1(\Omega) \cap C(\Omega, \mathbb{R}_+^*) \right) \times \mathcal{P}_c(\Omega)$ is an equilibrium if

$$\int_\Omega G_{\sigma,i}(x, w) d\mu(x) = L_i(w_i), \quad \forall i \in \{1, \dots, N\} \tag{32}$$

$$S_\theta(R_\sigma(x, w), Q(x)) \mu(x) = \psi(x), \quad for\ \mu - a.e.\ x \in \overline{\Omega} \tag{33}$$

$$-\Delta E(z) + \mathbf{V}(z) \cdot \nabla E(z) + \lambda E(z) = f_{\mu,w}(z), \quad \forall z \in \Omega \tag{34}$$

$$\operatorname{supp} \mu \subset \underset{x \in \overline{\Omega}}{\operatorname{argmax}}\ U_{\theta,\gamma}(R_\sigma(x, w), Q(x), E(x)) \tag{35}$$

completed with the condition $E = 0$ on $\partial\Omega$.

[5] We assume here that $p_k^N := 1/N$ for all $k \in \{1, \dots, N\}$.

We still assume that Assumptions 2.2 and 2.3 hold. The following Proposition gives existence and (conditions for) uniqueness of equilibria in this semi-discrete setting. The proof relies on a contraction argument.

Proposition 5.1. *There is at least one equilibrium in the sense of Definition 5.1. Moreover, there exists $\alpha_0 > 0$ and $\theta_0 > 0$ such that the equilibrium is unique for any $(\alpha_1, \alpha_2, \alpha_3, \theta) \in [0; \alpha_0]^3 \times [0; \theta_0]$.*

We can also prove, in this setting, the existence of solutions when there is no idiosyncratic preferences related to job locations, i.e. when the revenue of an agent is given by

$$R^N(x, w) = \max_{i \in \{1, \dots, N\}} \{w_i - c_i(x)\},$$

for all $x \in \overline{\Omega}$. Let

$$V_i(w) := \{x \in \overline{\Omega}, R^N(x, w) = w_i - c_i(x)\}$$

be the set of agents locations for which working for firm i is optimal, and, in case of multiple optimal choices, let

$$V_i^s(w) := V_i(w) \setminus \bigcup_{j \ne i} V_j(w)$$

be the set of agents locations for which i is strictly preferred to the other options. Thus, for all $i \in \{1, \dots, N\}$, the total labour supply at i lies in the interval $\left[\mu\left(V_i^s(w)\right), \mu\left(V_i(w)\right) \right]$, where $\mu \in \mathcal{P}_c(\Omega)$ is the equilibrium distribution of residents. The equilibrium condition in the labour market now writes

$$L_i(w_i) \in \left[\mu\left(V_i^s(w)\right), \mu\left(V_i(w)\right) \right], \quad \forall i \in \{1, \dots, N\} \tag{36}$$

where $L_i(w_i)$ is the labour demand at location i. Moreover, workers who are not hired by any firm are those for which w_0 is optimal:

$$1 - \sum_{i=1}^{N} L_i(w_i) \in \left[\mu\left(V_0^s(w)\right), \mu\left(V_0(w)\right) \right] \tag{37}$$

The other equilibrium conditions remain unchanged. Proposition 5.2 establishes existence of equilibria in this case.

Proposition 5.2. *There exists at least one quadruplet $(w, Q, E, \mu) \in (0, +\infty)^N \times C(\Omega, \mathbb{R}_+^*) \times \left(H_0^1(\Omega) \cap C(\Omega, \mathbb{R}_+^*) \right) \times \mathcal{P}_c(\Omega)$ satisfying conditions (33), (34), (35), (36) and (37), completed with $E = 0$ on $\partial\Omega$.*

6. Properties of the equilibria

In this section, we aim to explore some analytical aspects of the model. First, we demonstrate that when the agents are indifferent to pollution, the equilibrium is a Pareto one. Then, we analyse the effect of wind on the equilibrium.

6.1. Pareto efficiency

At the equilibrium, the Nash distribution of residents solves the mean-field problem

$$\sup_{m \in \mathcal{P}_c(\Omega)} \int_\Omega U_{\theta,\gamma}(R_\sigma(x, w), Q(x), \tilde{E}(x)) dm(x). \tag{38}$$

On the other hand, the Pareto-optimal distribution of residents should solve

$$\sup_{m \in \mathcal{P}_c(\Omega)} \int_\Omega U_{\theta,\gamma}(R_\sigma(x, w), Q[m](x), \tilde{E}[m](x)) dm(x) \tag{39}$$

where, given a distribution of residents $m \in \mathcal{P}_c(\Omega)$, $Q[m]$ is the clearing rental price solution to (17), and $E[m]$ is the pollution concentration solution to (18).

Problems (38) and (39) do not coincide in general, because contrary to the former, the latter takes into account that at the equilibrium, the rental price Q and pollution concentration E depend on the distribution of residents m.

Proposition 6.1 below shows that the Nash distribution of residents also solves

$$\sup_{m\in\mathcal{P}_c(\Omega)}\int_\Omega U_{\theta,\gamma}(R_\sigma(x,w),Q[m](x),\tilde{E}(x))dm(x),\tag{40}$$

where in the criterion only the function Q depends on m (not $\tilde{E}$).

Proposition 6.1. *Fix any $\gamma\geq 0$ and $w\in C(\Omega,\mathbb{R}_+^*)$. If the triplet $(Q,E,\mu)\in C(\Omega,\mathbb{R}_+^*)\times\left(H_0^1(\Omega)\cap C(\Omega,\mathbb{R}_+^*)\right)\times\mathcal{P}_c(\Omega)$ is solution to (17), (18) and (19), then μ solves*

$$\sup_{m\in\mathcal{P}_c(\Omega)}\int_\Omega U_{\theta,\gamma}(R_\sigma(x,w),Q[m](x),\tilde{E}(x))dm(x)\tag{41}$$

where, for any distribution of residents $m\in\mathcal{P}_c(\Omega)$, $Q[m]\in C(\Omega,\mathbb{R}_+^*)$ is the clearing rental price solution to (17).

Thus, without pollution externality ($\gamma=0$), the clearing rental price leads to a Pareto equilibrium. Indeed, when $\gamma=0$, the utility function does not depend on the pollution argument. However, when $\gamma>0$, problem (38) does not coincide with (39). In this case, the residential market is inefficient, because the agents do not internalize the effects of their location decisions on air pollution.

6.2. Wind effect

We aim at estimating the overall amount of pollution released, i.e. the quantity

$$\int_\Omega E(x)\,dx,\tag{42}$$

and the pollution suffered by an average resident, i.e. the quantity

$$\int_\Omega \tilde{E}(x)\mu(x)\,dx.\tag{43}$$

The following Proposition shows that the overall amount of pollution released is proportional to the average commuting distance travelled by an agent.

Proposition 6.2. *At the equilibrium,*

$$\lambda\int_\Omega E(x)\,dx=\mathbb{E}\left[|X-Y|\right],\tag{44}$$

where the couple (X,Y) follows the joint distribution of density $m(x,y):=\mu(x)G_\sigma(x,y,w)$.

Remark. If we replace the homogeneous Dirichlet by a Neumann boundary condition, we have

$$\lambda\int_\Omega E(x)\,dx=\mathbb{E}\left[|X-Y|\right]-\int_{\partial\Omega}E(s)\mathbf{V}(x)\cdot\mathbf{n}\,ds,$$

where $\mathbf{n}$ is the unit vector normal to the boundary $\partial\Omega$. The additional term represents the pollution that is conveyed out of the domain by wind.

We now turn to the pollution suffered by an average resident.

Proposition 6.3. *Assume that $E_0=1$. At the equilibrium,*

$$\begin{aligned}\lambda&\int_\Omega \tilde{E}(x)\mu(x)\,dx\\ &=\lambda+\int_\Omega f_{\mu,w}(x)\,\mu(x)\,dx-\int_\Omega\nabla E(x)\cdot\nabla\mu(x)\,dx\\ &+\frac{\theta}{1-\theta-\gamma}\int_\Omega\left(\mathbf{V}(x)\cdot\frac{\nabla R_\sigma(x,w)}{R_\sigma(x,w)}\right)E(x)\,\mu(x)\,dx\\ &-\frac{\theta}{1-\theta-\gamma}\frac{\gamma}{1-\theta}\int_\Omega\left(\mathbf{V}(x)\cdot\frac{\nabla R_\sigma(x,w)}{R_\sigma(x,w)}\right)\left(\tilde{E}(x)^{\frac{\gamma}{1-\theta}}-1\right)\mu(x)\,dx.\end{aligned}$$

Remark. If we replace the homogeneous Dirichlet by an homogeneous Neumann boundary condition, we have

$$\begin{aligned}\lambda&\int_\Omega \tilde{E}(x)\mu(x)\,dx\\ &=\lambda+\int_\Omega f_{\mu,w}(x)\,\mu(x)\,dx-\int_\Omega\nabla E(x)\cdot\nabla\mu(x)\,dx\\ &+\frac{\theta}{1-\theta-\gamma}\int_\Omega\left(\mathbf{V}(x)\cdot\frac{\nabla R_\sigma(x,w)}{R_\sigma(x,w)}\right)E(x)\,\mu(x)\,dx\\ &-\frac{\theta}{1-\theta-\gamma}\frac{\gamma}{1-\theta}\int_\Omega\left(\mathbf{V}(x)\cdot\frac{\nabla R_\sigma(x,w)}{R_\sigma(x,w)}\right)\left(\tilde{E}(x)^{\frac{\gamma}{1-\theta}}-1\right)\mu(x)\,dx\\ &-\frac{1-\theta}{1-\theta-\gamma}\int_{\partial\Omega}E(s)\mu(s)\mathbf{V}(s)\cdot\mathbf{n}\,ds,\end{aligned}$$

where $\mathbf{n}$ is the unit vector normal to the boundary $\partial\Omega$. The additional term represents the pollution that is conveyed out of the domain by wind.

Proposition 6.3 has an interesting interpretation that combines both economic and meteorological factors. It says that the air quality experienced by an average resident can be decomposed into three terms: a *source term*, a *diffusion term* and an *advection term*.

The source term (second term on the right-hand side of the equation) refers to the pollution emitted by the cars at the resident's location, just out front her house. It depends on the automobile traffic at this location, and therefore, on where people work and live.

The diffusion term (third term on the right-hand side of the equation) refers to the movement of pollution from surrounding areas through the diffusion process. It occurs due to the concentration gradients present in the atmosphere, with pollution spreading from areas of high concentration to areas of low concentration. If the gradients of pollution (∇E) and residents ($\nabla\mu$) are oriented in the same direction ($\nabla E\cdot\nabla\mu\geq 0$), pollution diffuses from areas of high population density to areas with less population density, which tends to decrease the pollution experienced by residents. However, if the gradients of pollution and population density are in opposite directions ($\nabla E\cdot\nabla\mu\leq 0$), pollution diffuses from areas of low population density to areas with high population density, which tends to increase the pollution experienced by residents.

The advection terms (two last terms on the right-hand side of the equation) refer to the movement of pollution by wind. It has an effect on the pollution suffered by the residents, depending on the relative orientation of wind ($\mathbf{V}$) and revenue gradient ($\nabla R_\sigma(\cdot,w)$). To simplify, let us assume that $\gamma=0$. This means that agents do not take pollution into account when choosing their place of residence, and merely prefer to live in proximity to regions with high revenues. If $\mathbf{V}\cdot\nabla R_\sigma(\cdot,w)$ is positive, the wind carries pollution towards areas with elevated revenue levels, where people tend to concentrate, resulting in increased pollution for residents. On the other hand, if $\mathbf{V}\cdot\nabla R_\sigma(\cdot,w)$ is negative, the wind carries pollution away from high revenue areas, reducing the pollution experienced by residents.

7. Numerical simulations

7.1. Algorithm

In Section 4, we characterized an equilibrium as a fixed point of a specific map. To numerically compute this equilibrium, we employ the Banach fixed-point iterative method. We have indeed demonstrated, through a contraction argument, that uniqueness is ensured when there is a finite number of workplaces and specific parameters remain small. In practice, we observe that the limit point computed is the same for several initial conditions. This suggests that uniqueness holds even though we have not proved it in the general case.

At each step, given a wage function w and pollution distribution E, the algorithm proceeds as follows:

(1) From w and E, it computes the distribution μ with (28). Then a new wage function w^* is computed clearing the labour market (16);
(2) From w, and μ computed in step 1, it computes a new pollution distribution E^* solution to (18);
(3) Finally, it computes the residual $r = \|w^* - w\|_{L^\infty} + \|E^* - E\|_{L^2}$.

The algorithm iterates as long as r is greater than a tolerance $\mathrm{tol} > 0$. Therefore, when the stopping criterion is satisfied, at the output of the loop, we get a fixed-point, i.e. an equilibrium characterized by w and E. The rental price function Q and the population distribution μ are then respectively recovered from Eqs. (17) and (28).

ALGORITHM 1: Equilibrium computation

Initialize: $E \leftarrow E^{(0)}$, $w \leftarrow w^{(0)}$, $r \leftarrow 1$

while $r > \mathrm{tol}$

1. Wage update: compute w^* as the unique solution to

$$\int_\Omega G_\sigma(x,y,w^*)\mu_{w^*,E}(x)\,dx = L(y,w^*(y)), \quad \forall y \in \overline{\Omega},$$

where $\mu_{w^*,E}$ is given by (28)

2. Pollution update: compute E^* as the unique solution to

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = f_{w,E}(z), & \forall z \in \Omega, \\ u(s) = 0, \quad \forall s \in \partial\Omega. \end{cases} \tag{45}$$

3. Residual and new values update: $r \leftarrow \|w^* - w\|_{L^\infty} + \|E^* - E\|_{L^2}$; $w \leftarrow w^*$; $E \leftarrow E^*$

end while

Compute μ with

$$\mu(x) = \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\,dy}, \quad \forall x \in \overline{\Omega}. \tag{46}$$

Compute Q with

$$Q(x) = (1-\theta)R_\sigma(x,w)\mu(x), \quad \forall x \in \overline{\Omega}.$$

Output: w, Q, E, μ.

7.2. Methods

Spatial discretization. For writing convenience, we focus on the case where $\overline{\Omega} = [0,1]$. Let X_h be a uniform grid on $\overline{\Omega}$ with step $h := 1/N_h$, $N_h \in \mathbb{N}^*$. The points of the grids are denoted by $x_j := jh$, for $j = 0,\dots,N_h$. The same grid is used to approximate both labour Eq. (16) and pollution dispersion Eq. (18). It would make sense to use a finer one for the dispersion equation, but this would involve further numerical complications that we prefer leaving for future research. The wage, rental price, pollution and residents distribution take the form of N_h-uplets (w_i), (Q_i), (E_i) and (μ_i), belonging to $(0;+\infty)^{N_h}$. All the integrals are approximated with the rectangle rule.

Economic equilibrium. Labour Eq. (16) is discretized as follows:

$$\sum_{i=0}^{N_h-1} h\,G_\sigma(x_i,x_j,w)\,\mu_i = L(x_j,w_j), \quad \forall j \in \{0,\dots,N_h\}, \tag{47}$$

where

$$G_\sigma(x_i,x_j,w) = \frac{e^{\frac{w_j - c(x_i,x_j)}{\sigma}}}{\sum_{k=0}^{N_h-1} h\,e^{\frac{w_k - c(x_i,x_k)}{\sigma}}},$$

$$\mu_i = \frac{R_\sigma(x_i,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}_i^{-\frac{\gamma}{1-\theta}}}{\sum_{k=0}^{N_h-1} R_\sigma(x_k,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}_k^{-\frac{\gamma}{1-\theta}}},$$

$$R_\sigma(x_i,w) = \sigma\ln\left(\sum_{k=0}^{N_h-1} e^{\frac{w_k - c(x_i,x_k)}{\sigma}}\right).$$

To solve the nonlinear system (47), we use the method *scipy.optimize.root* contained in the library "Scipy" of Python, which is based on the Powell hybrid method (Powell, 1970).

Pollution dispersion. To obtain a solution for (18), our strategy is to numerically simulate the stationary solution to

$$\begin{cases} \partial_t u(z,t) - \Delta u(z,t) + \mathbf{V}(z)\cdot\nabla u(z,t) + \lambda u(z,t) = f_{w,E}(z), & \forall (z,t) \in \Omega\times\mathbb{R}_+, \\ u(s,t) = 0, & \forall (s,t) \in \partial\Omega\times\mathbb{R}_+, \\ u(z,0) = E(z), & \forall z \in \Omega. \end{cases} \tag{48}$$

Let $\tau > 0$ be the time step. The solution u is discretized in time and space, such that $u_{j,n} = u(jh, n\tau)$, for all $(j,n) \in \{0,\dots,N_h\}\times\mathbb{N}$. To approximate Eq. (48), we use an explicit finite-difference scheme, i.e.

$$(\partial_t u)_{j,n} \approx \frac{u_{j,n+1} - u_{j,n}}{\tau}, \qquad (\nabla u)_{j,n} \approx \frac{u_{j+1,n} - u_{j-1,n}}{2h},$$
$$(\Delta u)_{j,n} \approx \frac{u_{j-1,n} - 2u_{j,n} + u_{j+1,n}}{h^2}.$$

Thus, the scheme takes the following form

$$\frac{u_{j,n+1} - u_{j,n}}{\tau} = \frac{u_{j-1,n} - 2u_{j,n} + u_{j+1,n}}{h^2} - V_j\frac{u_{j+1,n} - u_{j-1,n}}{2h} - \lambda u_{j,n} + f_{w,E,j},$$

with the following initial and boundary conditions:

- For all $j \in \{0,\dots,N_h\}$, $u_{j,0} = E_j$
- For all $n \in \mathbb{N}$, $u_{0,n} = 0$ and $u_{N_h,n} = 0$ (if homogeneous Dirichlet boundary conditions), or $u_{0,n} = u_{1,n}$ and $u_{N_h,n} = u_{N_h-1,n}$ (if homogeneous Neumann boundary conditions).[6]

This scheme is first-order accurate in time, and second-order in space. To ensure stability, the Courant–Friedrichs–Lewy condition must hold: $V\tau/h \le 1$.

7.3. Parameters

In the following, we assume that the production function is $F(l) = l^\beta$, where $\beta \in [0,1]$ and $l \ge 0$ is the labour factor. We focus on automobile-related pollution and neglect that stemming from heating and output production.[7] The density of firms $\nu(y)$ can take on many forms. In the context of a monocentric city, there is typically one Central Business District (CBD), and the density of firms decreases as the distance from the CBD increases: in the first case, that we call "classic monocentric city", the CBD is located in the geographical centre of the city; in the second case, called "shifted monocentric city", it is located in the west of the city (Fig. 2). Finally, we assume that the transportation cost is linear: $c(x,y) = c_0\|x-y\|$, for all $x,y \in \overline{\Omega}$.

All our baseline parameters are given in Table 1 (see Fig. 3).

7.4. Simulations

7.4.1. The role of pollution aversion

In this simulation, we use the "classic monocentric city" case (Fig. 2, left). We make the assumption that there is no wind effect, but we

[6] In the following simulations, we will indeed use Neumann (instead of Dirichlet) boundary conditions, in order to obtain more realistic numerical results.

[7] In Paris, in 2019, road transport was the main source (53%) of nitrogen oxide emissions (AirParif, 2019).

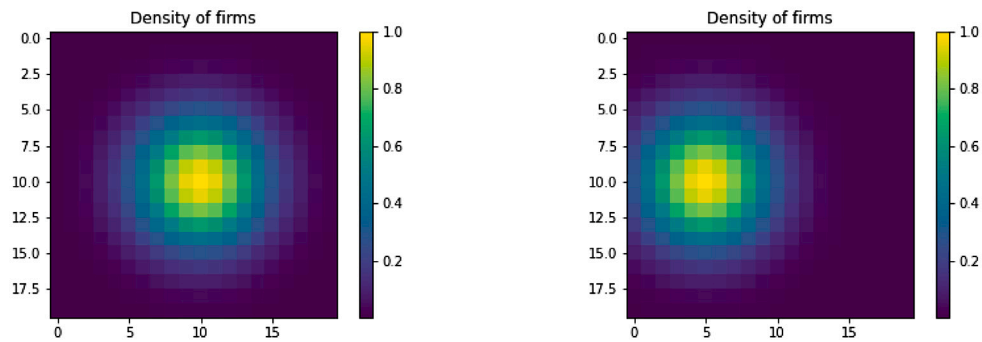

Fig. 2. Spatial concentration of firms in the 2D "classic monocentric city" (left), and the 2D "shifted monocentric city" (right).

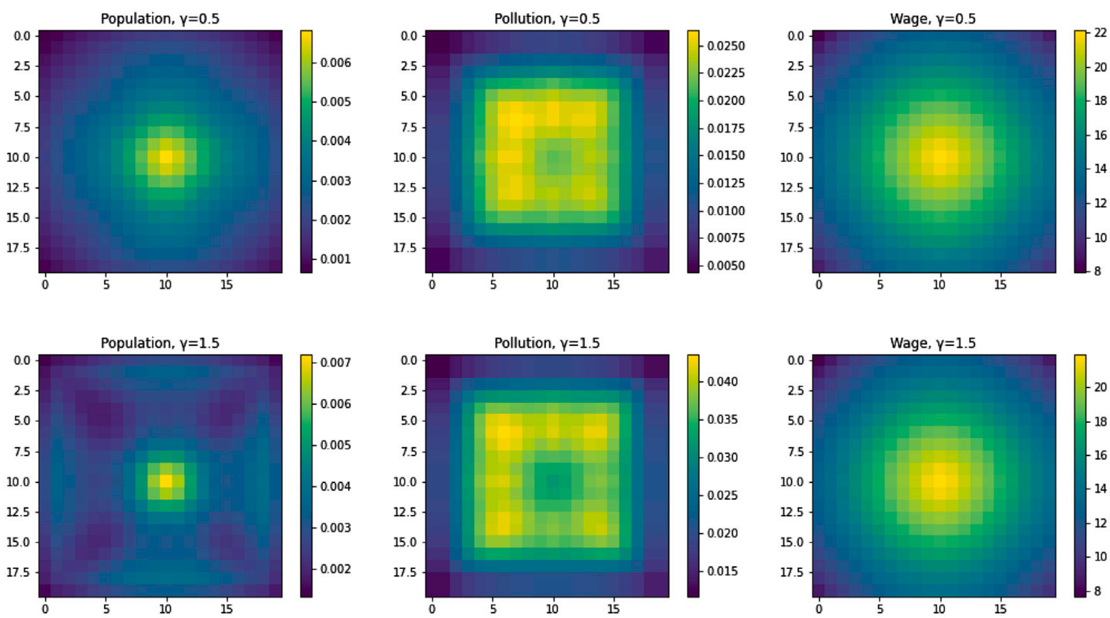

Fig. 3. Equilibrium in population, pollution and wage in the 2D classic monocentric city, for $\gamma = 0.5$ (top) and $\gamma = 1.5$ (bottom).

Table 1
Baseline parameters.

Parameter	Symbol	Value
Domain	$\overline{\Omega}$	$[0;20] \times [0;20]$
Consumption-housing substitution	θ	0.75
Aversion to pollution	γ	0.5
Transportation costs	c_0	1.0
Noise on effective salaries	σ	0.15
Capital–labour substitution	β	0.7
Number of firms	ν_0	1.0
Background pollution level	E_0	0.1
Wind	$\mathbf{V}$	[0.0,0.0]
Diffusion	k	1.0
Lessivage	λ	1.0
Coefficient source term	α	1.0
Discretization space step	h_x and h_y	$h_x = 1.0,\ h_y = 1.0$
Discretization time step	τ	0.1
Numerical tolerance	tol	0.05

do consider diffusion. We assume homogeneous Neumann boundary conditions on pollution. Fig. 3 displays the numerical results.

The initial situation depicted in Fig. 3 shows that when agents are less sensitive to pollution (γ is small), population density is higher close to the city centre. However, the pollution distribution does not reach its peak in the city centre; instead, it is highest in an area between the central business district (CBD) and the periphery. This is because a lot of commuting takes place in this particular area. More precisely, individuals living outside the city centre often opt for jobs near the CBD, leading to increased commuting and consequently elevated pollution levels in this region. On the contrary, there is not much commuting happening within the city centre itself, because residents from this zone work close to where they live, and there is no significant movement across the centre to reach the other side.

When we raise individuals' responsiveness to pollution (higher γ), we notice a couple of things. First, the wage distribution remains relatively unchanged, indicating that workers do not alter their workplace choices. Second, we see that workers leave the most polluted zone, namely the intermediate area, to live in the periphery and, for a small portion of them, in the city centre. With more agents living in the periphery, there is more commuting and therefore more pollution (Fig. 4).

7.4.2. The role of wind

We conduct numerical simulations in the 2D case to illustrate the role of wind. The city is assumed to have a monocentric structure, but with the business district shifted to the West (Fig. 2, right). We consider two types of wind, constant across the city: a West-East wind with velocity of $\mathbf{V} = (4.5;0)$ and an East-West wind with velocity of $\mathbf{V} = (-4.5;0)$. We set $\gamma = 1.5$, and other parameters as given in Table 1. We assume homogeneous Neumann boundary conditions on pollution.

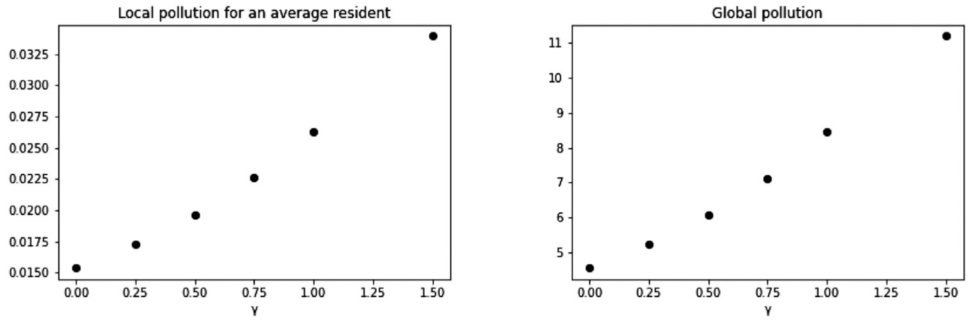

Fig. 4. Local and global pollution in the monocentric city, for different values of γ. Local pollution refers to the integral $\int_\Omega \tilde{E}\mu$, and global pollution to the integral $\int_\Omega \tilde{E}$.

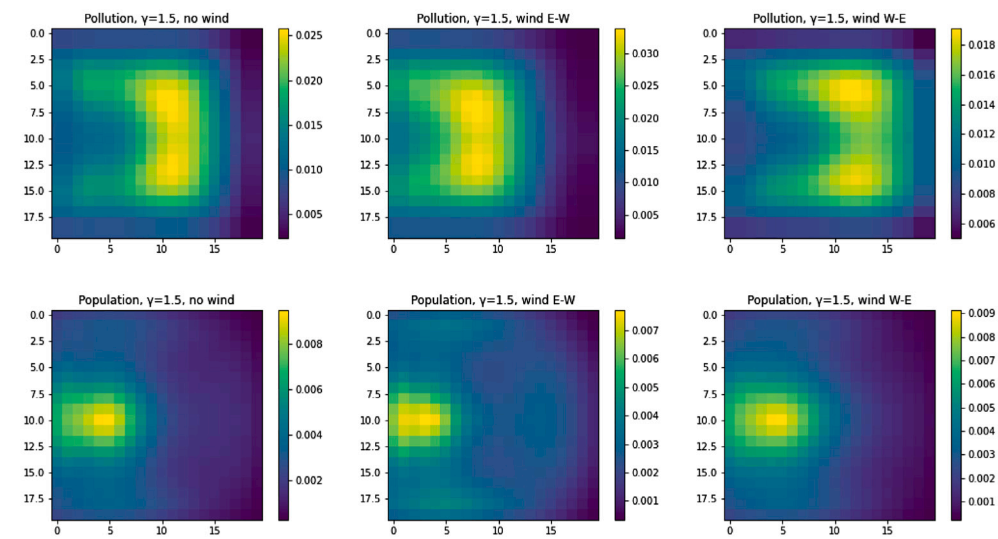

Fig. 5. Equilibrium in population and pollution in the "shifted" monocentric city, $\gamma = 1.5$, without wind (left), with an East-West wind (middle), and with a West-East wind (right).

Fig. 5 displays the resulting equilibria in pollution and population for the different wind regimes.

When the wind blows from an East-West direction (Fig. 5, middle), it pushes pollution towards the business district. In this case, the direction of the wind aligns with the direction of the revenue gradient. As a result, we see that people tend to live farther away from the business district, leading to increased overall pollution levels. This also contributes to increase the contamination suffered by an average resident (Fig. 6). For the same reasons as mentioned in Section 7.4.1, we also observe the presence of a middle area, between the business district and the periphery, which is more polluted and less densely populated.

When the wind blows from a West-East direction (Fig. 5, right), it pushes pollution away from the business district. In this case, the direction of the wind is opposite to the direction of the revenue gradient. As a result, people tend to live closer to the business district, leading to decreased overall pollution levels. This also contributes to reduce the contamination suffered by an average resident (Fig. 6).

We observe that the East-West wind has a greater impact on total pollution released compared to the West-East wind. This is because in the former case, the wind carries pollution to a more densely populated area, resulting in larger population movements to escape the pollution. This effect diminishes as the sensitivity of people to air pollution, γ, decreases. When $\gamma = 0$, the wind has no influence on the amount of pollution released.

8. Conclusion

We developed an equilibrium model of city in which the labour market, the residential market and pollution are interdependent. Our model differs from existing literature in that it allows for cities of any shape and includes a realistic description of pollution dispersion.

We proved existence (and, under additional assumptions, uniqueness) of equilibria, and we proposed a numerical method for computing solutions. We then examined various analytical and numerical applications of the model. In particular, we looked at the role of two parameters, pollution aversion and wind, on the equilibrium.

Our results emphasize the relevance of integrating physical and economic approaches in the study of urban air pollution. They open several avenues of research, such as investigating whether the equilibrium is unique, incorporating endogenous firm location and agglomeration externalities, examining the relationship between urban pollution and inequality through agent heterogeneity, and analysing regulatory issues such as the effects of a gasoline tax on the urban structure.

CRediT authorship contribution statement

Mohamed Bahlali: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Quentin Petit:**

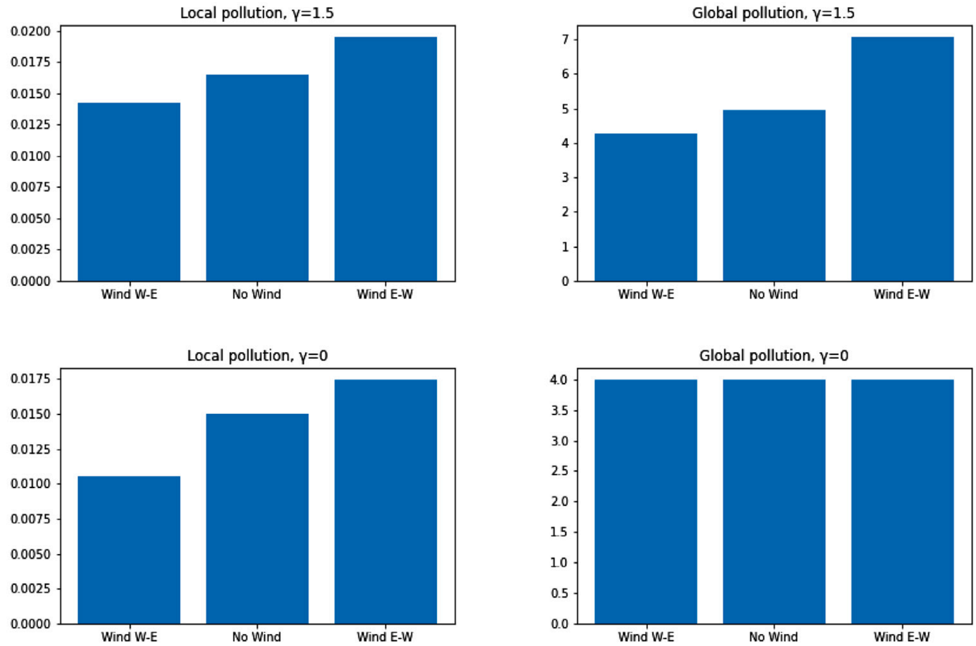

Fig. 6. Local and global pollution in the "shifted" monocentric city, for different wind regimes and values of γ. Local pollution refers to the integral $\int_\Omega \tilde{E}\mu + \frac{1-\theta}{1-\theta-\gamma}\int_{\partial\Omega} E\mu\mathbf{V}\cdot\mathbf{n}\,ds$, and global pollution to the integral $\int_\Omega f_{\mu,w}$.

Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare no known interests related to their submitted manuscript.

Data availability

No data was used for the research described in the article.

Acknowledgments

Mohamed Bahlali has received fundings from the EcoREES research collaboration between ENSAE, Université Paris-Dauphine, Ecole Polytechnique, Centrale-Supelec and EDF R&D, and the Low-carbon Innovation research program by the Climate Economics Chair.

Quentin Petit has received fundings from the Finance for Energy Market Research Centre between ENSAE, Université Paris-Dauphine, Ecole Polytechnique and EDF R&D; and the Finance and Sustainable Development Chair between Université Paris-Dauphine, Ecole Polytechnique, EDF R&D and Credit Agricole CIB.

We want to thank all of these initiatives for their financial support, and acknowledge very valuable comments from their respective teams. We would also like to thank the two anonymous referees for their valuable advice and constructive feedbacks.

Appendix. Proofs

On the convergence of R_σ to R when σ goes to zero

Proposition A.1. *For every $x \in \overline{\Omega}$,*

$$\sigma \ln\left(\int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}}\,dy\right) \to \max_{y\in\overline{\Omega}} w(y) - c(x,y)$$

when σ tends to 0.

Proof. Fix $x \in \overline{\Omega}$.

Step 1. We observe that

$$\sigma \ln\left(\int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}}\,dy\right) \leq \sigma \ln\left(\max_{y\in\overline{\Omega}} e^{\frac{w(y)-c(x,y)}{\sigma}}\right) = \max_{x\in\overline{\Omega}} w(y) - c(x,y).$$

Step 2. Let us fix $\varepsilon > 0$, and introduce

$$Y_\varepsilon = \{y \in \overline{\Omega} \ : \ w(y) - c(x,y) \geq \|w - c(x,\cdot)\|_\infty - \varepsilon\}.$$

Note that from the continuity of $w - c(x,\cdot)$, the Lebesgue measure of Y_ε is positive. We observe that

$$\begin{aligned}\sigma \ln\left(\int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}}\,dy\right) &\geq \sigma \ln\left(|Y_\varepsilon| e^{\frac{\|w-c(x,\cdot)\|_\infty-\varepsilon}{\sigma}}\right)\\ &= \|w - c(x,\cdot)\|_\infty - \varepsilon + \sigma \ln\left(|Y_\varepsilon|\right).\end{aligned}$$

Step 3. Finally, we get

$$\|w - c(x,\cdot)\|_\infty - \varepsilon + \sigma \ln\left(|Y_\varepsilon|\right) \leq \sigma \ln\left(\int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}}\,dy\right) \leq \|w - c(x,\cdot)\|_\infty.$$

Therefore,

$$\begin{aligned}\|w - c(x,\cdot)\|_\infty - \varepsilon &\leq \liminf_{\sigma\to 0} \sigma \ln\left(\int_\Omega e^{\frac{w(x)-c(x,y)}{\sigma}}\,dy\right)\\ &\leq \limsup_{\sigma\to 0} \sigma \ln\left(\int_\Omega e^{\frac{w(x)-c(x,y)}{\sigma}}\,dy\right) \leq \|w - c(\cdot,y)\|_\infty.\end{aligned}$$

Since it is true for an arbitrary $\varepsilon > 0$, we deduce that

$$\lim_{\sigma\to 0} \sigma \ln\left(\int_\Omega e^{\frac{w(y)-c(x,y)}{\sigma}}\,dy\right) = \|w - c(x,\cdot)\|_\infty. \quad \square$$

Proof of Proposition 3.1

By Lemma 4.1, we have

$$\mu(x) = \frac{R(x,w^*)^{\frac{\theta}{1-\theta}}\tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_0^1 R(y,w^*)^{\frac{\theta}{1-\theta}}\tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\,dy}, \quad \forall x \in [0,1]$$

Thus, by Eq. (23),

$$\tilde{E}(x) = E_0 + \lambda^{-1}\frac{\int_0^x R(s,w^*)^{\frac{\theta}{1-\theta}}\tilde{E}(s)^{-\frac{\gamma}{1-\theta}}\,ds}{\int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}}\tilde{E}(s)^{-\frac{\gamma}{1-\theta}}\,ds}$$

Differentiating w.r.t $x \in [0,1]$, we obtain the following differential equation

$$\tilde{E}'(x) = \lambda^{-1} \frac{R(x,w^*)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}} \tilde{E}(s)^{-\frac{\gamma}{1-\theta}} ds}$$

which, together with the boundary conditions $\tilde{E}(0) = E_0$ and $\tilde{E}(1) = E_0 + \lambda^{-1}$, admits the following unique solution

$$\tilde{E}(x) = \left[E_0^{\frac{1+\gamma-\theta}{1-\theta}} + \left(\left(E_0+\lambda^{-1}\right)^{\frac{1+\gamma-\theta}{1-\theta}} - E_0^{\frac{1+\gamma-\theta}{1-\theta}} \right) \frac{\int_0^x R(s,w^*)^{\frac{\theta}{1-\theta}} ds}{\int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}} ds} \right]^{\frac{1-\theta}{1+\gamma-\theta}}$$

Now, let $\beta := (1+\gamma-\theta)/(1-\theta)$, and $\varphi(x) := \int_0^x R(s,w^*)^{\frac{\theta}{1-\theta}} ds / \int_0^1 R(s,w^*)^{\frac{\theta}{1-\theta}} ds$. From (25), the derivative of $\tilde{E}(x)$ with respect to β has the same sign as

$$J(x) = \frac{\ln(E_0)E_0^\beta + \left(\ln(E_0+\lambda^{-1})(E_0+\lambda^{-1})^\beta - \ln(E_0)E_0^\beta\right)\varphi(x)}{E_0^\beta + \left((E_0+\lambda^{-1})^\beta - E_0^\beta\right)\varphi(x)} - \ln\left(E_0^\beta + \left((E_0+\lambda^{-1})^\beta - E_0^\beta\right)\varphi(x)\right)$$

We have $J(0) = (1-\beta)\ln(E_0)$, and for all $x \in [0,1]$, $J'(x)$ has the same sign as

$$\varphi'(x)\left[\ln(E_0+\lambda^{-1})(E_0+\lambda^{-1})^\beta - \ln(E_0)E_0^\beta - \frac{\ln(E_0)E_0^\beta + \left(\ln(E_0+\lambda^{-1})(E_0+\lambda^{-1})^\beta - \ln(E_0)E_0^\beta\right)\varphi(x)}{E_0^\beta + \left((E_0+\lambda^{-1})^\beta - E_0^\beta\right)\varphi(x)} - 1 \right]$$

which is equivalent, as λ goes to zero, to

$$\varphi'(x)\ln(\lambda^{-1})(\lambda^{-\beta} - 1)$$

$\varphi'(x)$ is positive, and the quantity $\ln(\lambda^{-1})(\lambda^{-\beta}-1)$ is also positive if λ is smaller than 1. Therefore, for all $x \in [0,1]$, $F(x) > 0$ and then $d\tilde{E}(x)/d\beta > 0$. As β increases with γ, this means that $d\tilde{E}(x)/d\gamma > 0$.

Proof of Lemma 4.1

If (w,Q,E,μ) is an equilibrium, then by Eqs. (2) and (17) we have, for all $x \in \overline{\Omega}$

$$U_{\theta,\gamma}(R_\sigma(x,w),\mu(x),\tilde{E}(x)) = \theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}\psi(x)^{1-\theta}}{\mu(x)^{1-\theta}}$$

Moreover, the mobility condition (19) is equivalent to: $\operatorname{supp}\mu \subset \operatorname{argmax}_{x\in\overline{\Omega}} U_{\theta,\gamma}(R_\sigma(x,w),\mu(x),E(x))$. This implies that there exists a real number β such that

$$\begin{cases} U_{\theta,\gamma}(R_\sigma(x,w),\mu(x),\tilde{E}(x)) \le \beta, & \forall x \in \overline{\Omega}, \\ U_{\theta,\gamma}(R_\sigma(x,w),\mu(x),\tilde{E}(x)) = \beta, & \forall x \in \operatorname{supp}\mu. \end{cases}$$

By (53), for all $x \in \overline{\Omega}$, $R_\sigma(x,w) \ge R(x,w) \ge w(x) - c(x,x) \ge \ell^{-1}(\eta)$. Then

$$\theta^\theta \frac{\ell^{-1}(\eta)^\theta \tilde{E}(x)^{-\gamma}\psi(x)^{1-\theta}}{\mu(x)^{1-\theta}} \le \theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}\psi(x)^{1-\theta}}{\mu(x)^{1-\theta}} \le \beta$$

This implies that for all $x \in \operatorname{supp}\mu$,

$$\theta^\theta \frac{\ell^{-1}(\eta)\ \|\tilde{E}\|_{L^\infty}^{-\gamma} \|\psi\|_{L^\infty}^{1-\theta}}{|\beta|+1} \le \mu(x)^{1-\theta}$$

This means that μ is bounded away from zero by a positive constant. By continuity of μ, we deduce that $\operatorname{supp}\mu = \overline{\Omega}$. Then

$$\theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}\psi(x)^{1-\theta}}{\mu(x)^{1-\theta}} = \beta, \quad \forall x \in \overline{\Omega}$$

Hence

$$\mu(x) = \left(\frac{\theta^\theta}{\beta}\right)^{\frac{1}{1-\theta}} R_\sigma(x,w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{\frac{-\gamma}{1-\theta}\psi(x)}$$

Since μ is a probability measure on $\overline{\Omega}$

$$\mu(x) = \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}} \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}\psi(x)}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}} \tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\psi(y)\,dy}, \quad \forall x \in \overline{\Omega}$$

Proof of Proposition 4.1

We first need to prove the following Lemma, which gives a regularity result about the equilibrium wage maps.

Lemma A.1. *Fix any $\mu \in \mathcal{P}_c(\Omega)$. If $w \in C(\Omega,\mathbb{R}_+^*)$ is a solution to (16), then $w \in C^1(\Omega,\mathbb{R}_+^*)$. Moreover, there exists $C_1, C_2 > 0$ independent of μ such that for all $y \in \Omega$*

$$|\nabla w(y)| \le e^{\|w\|_{L^\infty}}(C_1\ell(w(y)) + C_2) \tag{49}$$

Proof. If $w \in C(\Omega,\mathbb{R}_+^*)$ is a solution to (16), then for all $y \in \overline{\Omega}$

$$\frac{e^{\frac{w(y)}{\sigma}}}{\ell(w(y))} = \left(\int_\Omega \frac{e^{\frac{-c(x,y)}{\sigma}}}{\int_\Omega e^{\frac{w(z)-c(x,z)}{\sigma}} dz} d\mu(x)\right)^{-1} \nu(y) \tag{50}$$

Let

$$h(v) := \frac{e^{\frac{v}{\sigma}}}{\ell(v)}$$

The function h is a smooth bijection from $\mathbb{R}_+^*$ to $\mathbb{R}_+$. Using Eq. (50), we have, for all $y \in \overline{\Omega}$

$$w(y) = h^{-1}\left[\left(\int_\Omega \frac{e^{\frac{-c(x,y)}{\sigma}}}{\int_\Omega e^{\frac{w(z)-c(x,z)}{\sigma}} dz} d\mu(x)\right)^{-1} \nu(y)\right]$$

which shows that w is differentiable because h^{-1}, $c(x,\cdot)$ and ν are.

Now, if we differentiate Eq. (16), we get, for $y \in \overline{\Omega}$

$$\int_\Omega \nabla_y G_\sigma(x,y,w)d\mu(x) = \nabla_y L(y,w(y)) + \partial_v L(y,w(y))\nabla w(y)$$

i.e.

$$\int_\Omega \frac{G_\sigma(x,y,w)}{\sigma}(\nabla w(y) - \nabla_y c(x,y))\,d\mu(x) = \nabla_y L(y,w(y)) + \partial_v L(y,w(y))\nabla w(y)$$

For all $y \in \overline{\Omega}$, $\partial_v L(y,w(y)) < 0$, then $\int_\Omega \frac{G_\sigma(x,y,w)}{\sigma} d\mu(x) - \partial_v L(y,w(y)) > 0$, and therefore

$$\nabla w(y) = \frac{\nabla_y L(y,w(y)) + \int_\Omega \frac{G_\sigma(x,y,w)\nabla_y c(x,y)}{\sigma}}{\int_\Omega \frac{G_\sigma(x,y,w)}{\sigma} d\mu(x) - \partial_v L(y,w(y))}$$

We have

$$\left|\nabla_y L(y,w(y)) + \int_\Omega \frac{G_\sigma(x,y,w)\nabla_y c(x,y)}{\sigma}\right| \le \|\nabla\nu\|_{L^\infty}\ell(w(y)) + \sigma^{-1}\|\nabla_y c\|_{L^\infty}$$

and

$$\left|\int_\Omega \frac{G_\sigma(x,y,w)}{\sigma} d\mu(x) - \partial_v L(y,w(y))\right| \ge \int_\Omega \frac{G_\sigma(x,y,w)}{\sigma} d\mu(x) \ge \frac{e^{-\frac{\|w\|_{L^\infty}+\|c\|_{L^\infty}}{\sigma}}}{\sigma|\Omega|}$$

Then, for all $y \in \overline{\Omega}$

$$|\nabla w(y)| \le \sigma|\Omega| e^{\frac{\|w\|_{L^\infty}+\|c\|_{L^\infty}}{\sigma}} (\|\nabla\nu\|_{L^\infty}\ell(w(y)) + \sigma^{-1}\|\nabla_y c\|_{L^\infty})$$

which gives the desired estimate. □

Lemma A.1 shows that wage maps which are solutions to (16) belong to the following subset of $C(\overline{\Omega},\mathbb{R}_+^*)$

$$K_0 := \left\{ z \in C^1(\Omega,\mathbb{R}_+^*),\ |\nabla z(y)| \le e^{\|z\|_{L^\infty}}(C_1\ell(z(y)) + C_2) \quad \forall y \in \overline{\Omega} \right\}$$

where C_1 and C_2 are given in Lemma A.1. The subset K_0 is non empty (it contains the subset of constant and positive functions). Without loss of generality we can also assume that the solutions belong to the interior of K_0 (if not the case, expand the subset by taking an arbitrarily larger constant C_2).

Now, let $\varepsilon > 0$, and consider the map $\Lambda_\mu : \mathring{K}_0 \to \mathbb{R}$ defined by

$$\Lambda_\mu(z) = \phi_\mu(z) - \int_\Omega \int_\varepsilon^{z(y)} L(y,s)dsdy$$

where $\phi_\mu(z) = \int_\Omega R_\sigma(x,z)d\mu(x)$. The map Λ_μ is continuous on $(\mathring{K}_0, \|\cdot\|_\infty)$ and strictly convex. Moreover, for all $z \in \mathring{K}_0$

$$\Lambda_\mu(z) = \int_\Omega R_\sigma(x,z)d\mu(x) + \int_\Omega \nu(y)\pi(z(y))dy$$

because $L(y,s) = -\pi'(s)\nu(y)$. To show existence of a minimizer, we will provide a priori bounds on the solution and its derivative to reduce the minimization problem to a compact subset of $\mathring{K}_0$.

First a priori bound. Let fix z and $\hat{z}$ two elements of $\mathring{K}_0$ such that

$$\Lambda_\mu(z) \le \Lambda_\mu(\hat{z})$$

We note that

$$\Lambda_\mu(z) = \phi_\mu(z) - \int_{y\in\overline{\Omega}} \int_{s=\varepsilon}^{z(y)} L(y,s)\,ds\,dy \ge \|z\|_\infty - \|c\|_\infty - \int_\Omega \int_\varepsilon^{z(y)} L(y,s)\,ds\,dy$$

since $R_\sigma(x,z) \ge R(x,z) \ge \|z\|_\infty - \|c\|_\infty$. Then

$$\Lambda_\mu(\hat{z}) \ge \|z\|_\infty - \|c\|_\infty - \int_\Omega \int_\varepsilon^{z(y)} L(y,s)\,ds\,dy$$

and

$$\|z\|_\infty \le \|c\|_\infty + \phi_\mu(\hat{z}) + \int_\Omega \int_{\hat{z}(y)}^{z(y)} L(y,s)\,ds\,dy$$

Now

$$\phi_\mu(\hat{z}) \le \|\hat{z}\|_\infty + \sigma\ln(2)$$

because $R_\sigma(x,\hat{z}) \le R(x,\hat{z}) + \sigma\ln(2)$ and $R(x,\hat{z}) \le \|\hat{z}\|_\infty$. Besides, due to the monotonicity and positivity of the functions $L(y,\cdot)$, we have

$$\int_{\hat{z}(y)}^{z(y)} L(y,s)\,ds \ge L(y,\hat{z}(y))(z(y)-\hat{z}(y)) \ge L(y,\hat{z}(y))z(y)$$

Finally

$$\|z\|_\infty \le \|c\|_\infty + \|\hat{z}\|_\infty + \sigma\ln(2) + \|z\|_\infty \int_\Omega L(y,\hat{z}(y))\,dy$$

Thus, if z and $\hat{z}$ belong to $\mathring{K}_0$ and satisfy

$$\begin{cases} \Lambda_\mu(z) \le \Lambda_\mu(\hat{z}) \\ \int_\Omega L(y,\hat{z}(y))\,dy < 1 \end{cases}$$

we obtain a similar upper bound as in Petit (2022)

$$\|z\|_\infty \le \frac{\|c\|_\infty + \|\hat{z}\|_\infty + \sigma\ln(2)}{1 - \int_\Omega L(y,\hat{z}(y))\,dy}$$

As a consequence, if z is a minimizer of problem (26), we have

$$\|z\|_{L^\infty} \le M_1 \tag{51}$$

where

$$M_1 = \inf\left\{ \frac{\|c\|_\infty + \|\hat{z}\|_\infty + \sigma\ln(2)}{1 - \int_\Omega L(y,\hat{z}(y))\,dy}, \hat{z} \in \mathring{K}_0, L(y,\hat{z}(y))\,dy < 1 \right\} \tag{52}$$

Second a priori bound. We are now looking for a bound from below. We claim that if $z \in \mathring{K}_0$ is a minimizer, then for all $y \in \overline{\Omega}$, $L(y,z(y)) \le 1$, i.e. $z(y) \ge \ell^{-1}(\nu(y)^{-1})$. For $z \in \mathring{K}_0$, the Fréchet derivative of Λ_μ at z is the following application

$$D_z\Lambda_\mu : \mathring{K}_0 \to \mathbb{R},\ h \mapsto \int_{\Omega^2} G_\sigma(x,y,z)h(y)\,dy\,d\mu(x) - \int_\Omega L(y,z(y))h(y)\,dy$$

Assume by contradiction that there exists $y^* \in \overline{\Omega}$, $L(y^*, z(y^*)) > 1$. As L and z are continuous, there exists an open ball $B_r(y^*) \subset \Omega$ with $r > 0$, such that for all $y \in B_r(y^*)$, $L(y,z(y)) > 1$. Let $h^0 \in \mathring{K}_0$ and such that $h^0_{|B_r(y^*)} > 0$ and $h^0_{|\overline{\Omega}\setminus B_r(y^*)} = 0$. Then

$$\int_\Omega L(y,z(y))h^0(y)\,dy = \int_{B_r(y^*)} L(y,z(y))h^0(y)\,dy > \int_{B_r(y^*)} h^0(y)\,dy$$

and

$$\begin{aligned}\int_{\Omega^2} G_\sigma(x,y,z)h^0(y)\,dy\,d\mu(x) &= \int_{\overline{\Omega}\times B_r(y^*)} G_\sigma(x,y,z)h^0(y)\,dy\,d\mu(x) \\ &\le \int_{\overline{\Omega}\times B_r(y^*)} h^0(y)\,dy\,d\mu(x) \\ &\le \int_{B_r(y^*)} h^0(y)\,dy\end{aligned}$$

Therefore

$$D_z\Lambda_\mu.h^0 < 0$$

which means that in this case, z is not a minimizer. Conclusion: if $z \in \mathring{K}_0$ is a minimizer, then

$$\forall y \in \overline{\Omega},\quad \ell(z(y)) \le \nu(y)^{-1} \le \eta \tag{53}$$

Third a priori bound. Now, if $z \in K_0$ satisfies the a priori bounds (51) and (53), then, by inequality (49)

$$\|\nabla z\|_{L^\infty} \le e^{M_1}(C_1\eta + C_2)$$

meaning that we have a constant $M_2 > 0$ such that

$$\|\nabla z\|_{L^\infty} \le M_2$$

Let us introduce the following subset of $\mathring{K}_0$

$$K_1 := \left\{ z \in C^1(\Omega,\mathbb{R}_+),\ z(\cdot) \ge \ell^{-1}(\eta),\ \|z\|_{L^\infty} \le M_1,\ \|\nabla z\|_{L^\infty} \le M_2 \right\}$$

The subset K_1 is convex and compact for the uniform norm $\|\cdot\|_{L^\infty}$, as a consequence of Ascoli–Arzelà theorem. We have proved that

$$\min_{z\in\mathring{K}_0} \Lambda_\mu(z) = \min_{z\in K_1} \Lambda_\mu(z)$$

Conclusion. Let us take a minimizing sequence $(w_n)_{n\in\mathbb{N}}$ of the problem

$$\min_{w\in K_1} \Lambda_\mu(w)$$

The compactness of K_1 and continuity of Λ_μ ensure the existence of a minimizer $w \in K_1$. The uniqueness is ensured by the strict convexity of Λ_μ. This provides the existence and uniqueness of a solution to (26).

Characterization of the minimizer. Since Λ_μ is strictly convex and smooth, $w \in C(\Omega,\mathbb{R}_+^*)$ is a minimizer if and only if $D_w\Lambda_\mu = 0$, i.e. for all $y \in \overline{\Omega}$

$$\int_\Omega G_\sigma(x,y,w)d\mu(x) = L(y,w(y))$$

Proof of Proposition 4.2

The proof follows (Achdou et al., 2023), subsection 3.1. Fix any $\mu \in \mathcal{P}_c(\Omega)$. By Proposition 4.1, if $w \in C(\Omega,\mathbb{R}_+^*)$ is solution to (16) then w is the unique minimizer of

$$\min_{w\in C(\Omega)} \left\{ \int_\Omega \nu(y)\left[F(\ell(w(y))) - \ell(w(y))w(y)\right]\,dy + \int_\Omega R_\sigma(x,w)d\mu(x) + \mathbb{I}_{K_1}(w) \right\} \tag{54}$$

where $\mathbb{I}_{K_1}(w) = 0$ if $w \in K_1$, else $\mathbb{I}_{K_1}(w) = +\infty$. The Fenchel-Rockafellar dual problem writes

$$\sup_{l\in C(\Omega)} \left\{ \int_\Omega \nu(y)F\left(\nu(y)^{-1}l(y)\right)\,dy - C_\sigma(l) \right\} \tag{55}$$

where

$$C_\sigma(l) := \sup_{w\in C(\Omega)} \left\{ \int_\Omega l(y)w(y)\,dy - \int_\Omega R_\sigma(x,w)d\mu(x) - \mathbb{I}_{K_1}(w) \right\} \tag{56}$$

Problem (54) is a convex and continuous minimization problem. Thus, the supremum in (55) is attained for a certain $l^* \in C(\overline{\Omega})$ and strong duality holds:

$$\min_{w\in C(\Omega)} \left\{ \int_\Omega \nu(y)\left[F(\ell(w(y))) - \ell(w(y))w(y)\right]\,dy + \int_\Omega R_\sigma(x,w)d\mu(x) + \mathbb{I}_{K_1}(w) \right\}$$
$$= \max_{l\in C(\Omega)} \left\{ \int_\Omega \nu(y) f\left(\nu(y)^{-1}l(y)\right)\,dy - C_\sigma(l) \right\}$$

A necessary condition for $C_\sigma(l^*)$ not being equal to infinity is $l^* \geq 0$ and $\int_\Omega l^*(y)\,dy \leq 1$. In this case, optimality conditions for (56) yield

$$l^*(y) = \int_\Omega G_\sigma(x,y,w)\,d\mu(x), \quad \forall y \in \overline{\Omega}$$

Now, consider the map $\Theta : C(\Omega) \to C(\Omega)$ that associates, for any $w \in C(\Omega)$, the function l defined by $y \mapsto \int_\Omega G_\sigma(x,y,w)\,d\mu(x)$. By Proposition 4.1, Θ is a bijection. Thus problem (55) is equivalent to

$$\max_{l\in C(\Omega)} \left\{ \int_\Omega [\nu(y)F\left(\nu(y)^{-1}l(y)\right) - l(y)\Theta^{-1}(l)(y)]\,dy + \int_\Omega R_\sigma(x,\Theta^{-1}(l))d\mu(x) + \mathbb{I}_{K_1}(\Theta^{-1}(l)) \right\}$$

which is equivalent to

$$\max_{w\in K_1} \left\{ \int_\Omega [\nu(y)F\left(\nu(y)^{-1}\Theta(w)(y)\right) - \Theta(w)(y)w(y)]\,dy + \int_\Omega R_\sigma(x,w)d\mu(x) \right\}$$

i.e.

$$\max_{w\in K_1} \left\{ \int_\Omega [\nu(y)F\left(\nu(y)^{-1}l_{\mu,w}(y)\right) - l_{\mu,w}(y)w(y)]dy + \int_\Omega R_\sigma(x,w)d\mu(x) \right\}$$

Thus

$$\min_{w\in K_1} \left\{ \int_\Omega \nu(y)\left[F(\ell(w(y))) - \ell(w(y))w(y)\right]\,dy + \int_\Omega R_\sigma(x,w)d\mu(x) \right\}$$
$$= \max_{w\in K_1} \left\{ \int_\Omega [\nu(y)F\left(\nu(y)^{-1}l_{\mu,w}(y)\right) - l_{\mu,w}(y)w(y)]dy + \int_\Omega R_\sigma(x,w)d\mu(x) \right\} \tag{57}$$

Let w_0 be the unique solution to (16): for all $y \in \overline{\Omega}$, $\nu(y)\ell(w_0) = l_{\mu,w_0}(y)$. Hence

$$\int_\Omega \nu(y)\left[F(\ell(w_0(y))) - \ell(w_0(y))w_0(y)\right]\,dy + \int_\Omega R_\sigma(x,w_0)d\mu(x)$$
$$= \int_\Omega [\nu(y)F\left(\nu(y)^{-1}l_{\mu,w_0}(y)\right) - l_{\mu,w_0}(y)w_0(y)]dy + \int_\Omega R_\sigma(x,w_0)d\mu(x)$$

which, together with equality (57), ensures that w_0 achieves the supremum in (27).

Proof of Proposition 4.3

For the existence and uniqueness part, we apply Riesz's representation theorem. Let us consider the following inner product, on $H_0^1(\Omega)$

$$(u,v) := \int_\Omega \nabla u\cdot\nabla v + \int_\Omega (\mathbf{V}\cdot\nabla u)\,v + \lambda\int_\Omega uv$$

The positive definite property of this inner product is ensured by the fact that λ is positive, and

$$\int_\Omega (\mathbf{V}\cdot\nabla u)\,u = \int_\Omega \mathbf{V}\cdot\nabla\left(\frac{1}{2}u\cdot u\right)$$
$$= -\int_\Omega (\nabla\cdot\mathbf{V})\left(\frac{1}{2}u\cdot u\right) + \int_{\partial\Omega}\left(\frac{1}{2}u\cdot u\right)\mathbf{V}\cdot\mathbf{n}\,ds$$
$$= 0$$

where we first used the divergence theorem, and then the fact that $\nabla\cdot\mathbf{V}(z) = 0$ for all $z \in \Omega$, and $u(s) = 0$ for all $s \in \partial\Omega$ (see Section 2.4). Now, consider the linear functional

$$\Lambda : H_0^1(\Omega) \to \mathbb{R},\ v \mapsto \int_\Omega f_{w,q}v$$

Hölder inequality gives, for all $v \in H_0^1(\Omega)$

$$|\Lambda(v)| \leq \|f_{w,q}\|_{L^2}\,\|v\|_{L^2}$$

Thus, Λ is a bounded linear operator on $H_0^1(\Omega)$, thus a linear form on this Hilbert space. By Riesz's representation theorem, there exists a unique $u_{w,q} \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$, $\Lambda(v) = (u_{w,q}|v)$ i.e.

$$\int_\Omega \nabla u_{w,q}\cdot\nabla v + \int_\Omega (\mathbf{V}\cdot\nabla u)\,v + \lambda\int_\Omega u_{w,q}v = \int_\Omega f_{w,q}v \tag{58}$$

The positivity of $u_{w,q}$ is a direct consequence of the maximum principle. Regarding the majoration of $\nabla u_{w,q}$ in $L^2(\Omega)$, Eq. (58) applied to $v = u_{w,q}$ yields

$$\|\nabla u_{w,q}\|_{L^2}^2 + \lambda\|u_{w,q}\|_{L^2}^2 \leq \|f_{w,q}\|_{L^2}\,\|u_{w,q}\|_{L^2}$$

by Hölder inequality. Then

$$\min(1,\lambda)\|u_{w,q}\|_{H_0^1}^2 \leq \|f_{w,q}\|_{L^2}\,\|u_{w,q}\|_{H_0^1} \leq |\Omega|\,\|f_{w,q}\|_{L^\infty}^2\,\|u_{w,q}\|_{H_0^1}$$

where $\|v\|_{H_0^1} := \left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right)^{1/2}$ for all $v \in H_0^1(\Omega)$. Therefore

$$\|u_{w,q}\|_{H_0^1} \leq |\Omega|\,\delta^{-2}\min(1,\lambda)^{-1}$$

which yields

$$\|\nabla u_{w,q}\|_{L^2} \leq |\Omega|\,\delta^{-2}\min(1,\lambda)^{-1}$$

Convexity of K_2 is immediate. The proof of compactness is inspired by Le Dret (2013). Let us denote $k_0 := |\Omega|\,\delta^{-2}\min(1,\lambda)^{-1}$. By Rellich's theorem, the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is compact. Therefore K_2, which is bounded in $H_0^1(\Omega)$, is relatively compact in $L^2(\Omega)$. Let us show that K_2 is closed in $L^2(\Omega)$. If $(v_n) \in K_2^{\mathbb{N}}$ converges to $v \in L^2(\Omega)$, then (v_n) is bounded in $H_0^1(\Omega)$ and contains a subsequence $(v_{n'})$ that converges weakly to $v' \in H_0^1(\Omega)$. By uniqueness of the limit, $v' = v$, and the lower semicontinuity of the norm implies $\|v\|_{H^1} \leq \liminf_{n'\infty}\|v_{n'}\|_{H^1} \leq k_0$ (here we consider the semi-norm $\|v\|_{H_0^1} := \|\nabla v\|_{L^2}$ by Poincaré inequality). Consequently, $v \in K_2$, and K_2 is compact.

Proof of Proposition 4.5

Continuity of the map $\mathcal{Y}_1$ on $(K_1, \|\cdot\|_{L^\infty})$

We first need to prove the following two lemmas. The first one proves continuity of the equilibrium distribution of residents, explicitly given by (28), with respect to w and E. The second one proves (weak) continuity of the solutions to problem (26) with respect to μ.

Lemma A.2.

(1) *Let $E \in K_2$, and (w_n) be a sequence in K_1. If $\|w_n - w\|_{L^\infty} \to 0$ for some $w \in K_1$, then $\|\mu(w_n,E) - \mu(w,E)\|_{L^1} \to 0$.*

(2) *Let $w \in K_1$, and (E_n) be a sequence in K_2. If $\|E_n - E\|_{L^2} \to 0$ for some $E \in K_2$, then $\|\mu(w,E_n) - \mu(w,E)\|_{L^1} \to 0$.*

Proof.

(1) Let us denote $\mu_n := \mu(w_n,E)$ and $\mu := \mu(w,E)$. Let $x \in \Omega$. We have

$$|\mu_n(x) - \mu(x)|$$
$$= \left| \frac{R_\sigma(x,w_n)^{\frac{\theta}{1-\theta}}\,\tilde{E}(x)^{\frac{-\gamma}{1-\theta}}\,\psi(x)}{\int_\Omega R_\sigma(y,w_n)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\,\psi(y)\,dy} - \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(x)^{\frac{-\gamma}{1-\theta}}\,\psi(x)}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\,\psi(y)\,dy} \right|$$
$$= \left(\int_\Omega R_\sigma(y,w_n)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\,\psi(y)\,dy\right)^{-1}\left(\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\,\psi(y)\,dy\right)^{-1}$$

$$\cdot \left| \int_{\Omega} (\tilde{E}(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} \psi(x)\psi(y) \left[(R_\sigma(x,w_n)R_\sigma(y,w))^{\frac{\theta}{1-\theta}} - (R_\sigma(y,w_n)R_\sigma(x,w))^{\frac{\theta}{1-\theta}} \right] dy \right|$$

$$\leq C \int_{\Omega} \left| (R_\sigma(x,w_n)R_\sigma(y,w))^{\frac{\theta}{1-\theta}} - (R_\sigma(y,w_n)R_\sigma(x,w))^{\frac{\theta}{1-\theta}} \right| dy$$

for some constant $C > 0$, because $\tilde{E}$, ψ and $R_\sigma(\cdot, w)$ are bounded from below and above by positive constants. Especially, $R_\sigma(\cdot, w)$ is bounded from below by $R_- := \ell^{-1}(\eta)$, and from above by $R_+ := M_1 + \sigma \ln(2)$, with M_1 given by (52). The function $\mathbb{R} \ni a \mapsto a^{\frac{\theta}{1-\theta}}$ has continuous and bounded derivative on $[R_-^2; R_+^2]$. It is therefore Lipschitz on this segment, hence for all $y \in \Omega$,

$$\left| (R_\sigma(x,w_n)R_\sigma(y,w))^{\frac{\theta}{1-\theta}} - (R_\sigma(y,w_n)R_\sigma(x,w))^{\frac{\theta}{1-\theta}} \right| \leq C \left| R_\sigma(x,w_n)R_\sigma(y,w) - R_\sigma(y,w_n)R_\sigma(x,w) \right|$$

for some constant $C \geq 0$. Besides

$$\begin{aligned} &\left| R_\sigma(x,w_n)R_\sigma(y,w) - R_\sigma(y,w_n)R_\sigma(x,w) \right| \\ &\quad \leq (R_\sigma(y,w_n) + R_\sigma(y,w))|R_\sigma(x,w_n) - R_\sigma(x,w)| \\ &\quad \leq 2\,R_+\,|R_\sigma(x,w_n) - R_\sigma(x,w)| \\ &\quad \leq C\|w - w_n\|_{L^\infty} \end{aligned}$$

for another $C \geq 0$. The majoration comes from the fact that $\ln(\cdot)$ and $\exp(\cdot)$ are Lipschitz on compact subsets of, respectively, $\mathbb{R}_+^*$ and $\mathbb{R}$. Thus

$$|\mu_n(x) - \mu(x)| \leq C\|w - w_n\|_{L^\infty}$$

for another $C \geq 0$. This gives the L^1 convergence of (μ_n) to μ.

(2) Let us denote $\mu_n := \mu(w, E_n)$ and $\mu := \mu(w, E)$. For all $x \in \Omega$, we have

$$\begin{aligned} &|\mu_n(x) - \mu(x)| \\ &= \left| \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}_n(x)^{\frac{-\gamma}{1-\theta}}\psi(x)}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}_n(y)^{\frac{-\gamma}{1-\theta}}\psi(y)\,dy} - \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(x)^{\frac{-\gamma}{1-\theta}}\psi(x)}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\psi(y)\,dy} \right| \\ &= \left(\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}_n(y)^{\frac{-\gamma}{1-\theta}}\psi(y)\,dy \right)^{-1} \left(\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\,\tilde{E}(y)^{\frac{-\gamma}{1-\theta}}\psi(y)\,dy \right)^{-1} \\ &\quad \cdot \left| \int_\Omega (R_\sigma(x,w)R_\sigma(y,w))^{\frac{\theta}{1-\theta}}\psi(x)\psi(y) \left[(\tilde{E}_n(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} - (\tilde{E}_n(y)\tilde{E}(x))^{\frac{-\gamma}{1-\theta}} \right] dy \right| \\ &\leq C \int_\Omega \left| (\tilde{E}_n(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} - (\tilde{E}_n(y)\tilde{E}(x))^{\frac{-\gamma}{1-\theta}} \right| dy \end{aligned}$$

for some constant $C \geq 0$, because $\tilde{E}$, ψ and $R_\sigma(\cdot, w)$ are bounded from below and above by positive constants. Now, the function $\mathbb{R} \ni a \mapsto a^{\frac{-\gamma}{1-\theta}}$ has continuous and bounded derivative on $[E_0^2; +\infty)$ and is therefore Lipschitz on this interval, hence for all $x, y \in \Omega$,

$$\left| (\tilde{E}_n(x)\tilde{E}(y))^{\frac{-\gamma}{1-\theta}} - (\tilde{E}_n(y)\tilde{E}(x))^{\frac{-\gamma}{1-\theta}} \right| \leq C|\tilde{E}_n(x)\tilde{E}(y) - \tilde{E}_n(y)\tilde{E}(x)|$$

for another $C \geq 0$. Besides, for all $x, y \in \Omega$

$$|\tilde{E}_n(x)\tilde{E}(y) - \tilde{E}_n(y)\tilde{E}(x)| \leq \tilde{E}_n(x)|E_n(y) - E(y)| + \tilde{E}_n(y)|E_n(x) - E(x)|$$

Thus, for all $x, y \in \Omega$

$$|\mu_n(x) - \mu(x)| \leq C\left[\tilde{E}_n(x)|E_n(y) - E(y)| + \tilde{E}_n(y)|E_n(x) - E(x)|\right]$$

By integrating the previous inequality on Ω^2, and using Hölder inequality, we get

$$\|\mu_n - \mu\|_{L^1} \leq 2C|\Omega|\,\|\tilde{E}_n\|_{L^2}\,\|E_n - E\|_{L^2}$$

which gives the L^1 convergence of (μ_n) to μ.

Remark. We could simplify the proof by reasoning by composition. However, we opted for the current proof method because we needed Lipschitz estimates to prove the uniqueness result in Section 5. □

Lemma A.3. *Let (μ_n) be a sequence in $\mathcal{P}_c(\Omega)$ and (w_n) be the sequence of associated minimizers in (26). If $\mu_n \to \mu$ for the weak-$\star$ topology then (w_n) converges to w_0, the minimizer associated with μ, in $(K_1, \|\cdot\|_{L^\infty})$.*

Proof. For any $w \in K_1$ and $\mu_1, \mu_2 \in \mathcal{P}(\Omega)$, we have

$$\begin{aligned} |\Lambda_{\mu_1}(w) - \Lambda_{\mu_2}(w)| &= \left| \int_\Omega R_\sigma(x,w)(d\mu_1(x) - d\mu_2(x)) \right| \\ &\leq (\|w\|_{L^\infty} + \sigma\ln(2) + \|\nabla c\|_{L^\infty})d_1(\mu_1,\mu_2) \end{aligned}$$

which comes from the fact that the map $\Omega \ni x \mapsto R_\sigma(x, w)$ is uniformly bounded by $\|w\|_\infty + \sigma\ln(2)$ and is $\|\nabla c\|_{L^\infty}$-Lipschitz. Then by compactness of K_1,

$$\min_{K_1} \Lambda_{\mu_n} \to \min_{K_1} \Lambda_\mu$$

and there exists $\tilde{w} \in K_1$ such that, up to the extraction of a subsequence, $w_n \to \tilde{w}$ in $(K_1, \|\cdot\|_{L^\infty})$. Therefore

$$\begin{aligned} \left| \min_{K_1} \Lambda_{\mu_n} - \Lambda_\mu(\tilde{w}) \right| &= \left| \Lambda_{\mu_n}(w_n) - \Lambda_\mu(\tilde{w}) \right| \\ &\leq |\Lambda_{\mu_n}(w_n) - \Lambda_\mu(w_n)| + |\Lambda_\mu(w_n) - \Lambda_\mu(\tilde{w})| \\ &\leq (\|w_n\|_{L^\infty} + \sigma\ln(2) + \|\nabla c\|_{L^\infty})d_1(\mu_n,\mu) \\ &\quad + |\Lambda_\mu(w_n) - \Lambda_\mu(\tilde{w})| \end{aligned}$$

which goes to zero by continuity of Λ_μ. This ensures that $\min_{K_1} \Lambda_{\mu_n}$ converges to $\Lambda_\mu(\tilde{w})$ when n goes to $+\infty$. The uniqueness of the limit ensures that

$$\Lambda_\mu(\tilde{w}) = \min_{K_1} \Lambda_\mu$$

From Proposition 4.1, there is a unique solution to (26), namely w_0. Hence $\tilde{w} = w_0$.

Remark. We could simplify the proof by reasoning by composition. However, we opted for the current proof method because we needed Lipschitz estimates to prove the uniqueness result in Section 5. □

We are now able to prove continuity of $\mathcal{Y}_1$ on $(K_1, \|\cdot\|_{L^\infty})$.

If (w_n) converges to w in $(K_1, \|\cdot\|_{L^\infty})$, by Lemma A.2 $(\mu(w_n, E))$ converges to $\mu(w, E)$ in $L^1(\Omega)$, and therefore weakly converges to the same limit. Then, by Lemma A.3, $(\mathcal{Y}_1(w_n, E))$ uniformly converges to $\mathcal{Y}_1(w, E)$.

Similarly, if (E_n) converges to E in $(K_2, \|\cdot\|_{L^2})$, by Lemma A.2 $(\mu(w, E_n))$ converges to $\mu(w, E)$ in $L^1(\Omega)$, and therefore weakly converges to the same limit. Then, by Lemma A.3, $(\mathcal{Y}_1(w, E_n))$ uniformly converges to $\mathcal{Y}_1(w, E)$.

Continuity of the map $\mathcal{Y}_2$ on $(K_2, \|\cdot\|_{L^2})$

We first need to prove the following preliminary results, which show continuity of the source term $f_{w,E}$ with respect to w and E, and continuity of the solutions to the scalar transport equation with respect to the source term.

Lemma A.4.

(1) Let $E \in K_2$, and (w_n) be a sequence in K_1. If $\|w_n - w\|_{L^\infty} \to 0$ for some $w \in K_1$, then $\|f_{w_n,E} - f_{w,E}\|_{L^\infty} \to 0$.

(2) Let $w \in K_1$, and (E_n) be a sequence in K_2. If $\|E_n - E\|_{L^2} \to 0$ for some $E \in K_2$, then $\|f_{w,E_n} - f_{w,E}\|_{L^\infty} \to 0$.

Proof.

(1) Let us denote $f_n := f_{w_n,E}$, $f := f_{w,E}$, $\mu_n := \mu(w_n, E)$ and $\mu := \mu(w, E)$. Let $z \in \overline{\Omega}$. We have

$$\begin{aligned} |f_n(z) - f(z)| &\leq \alpha_2 v(z)|F(\ell(w_n(z))) - F(\ell(w(z)))| \\ &\quad + \alpha_3 \int_{\Omega^2} \delta^{-1}\left|G_\sigma(x,y,w_n)\mu_n(x)\right. \end{aligned}$$

$$\begin{aligned}
&- G_\sigma(x,y,w)\mu(x)\Big| dx dy \\
&\le \alpha_2 C \|w_n - w\|_{L^\infty} \\
&+ \alpha_3 \int_{\Omega^2} \delta^{-1} \left|\mu_n(x) - \mu(x)\right| G_\sigma(x,y,w_n)\, dx dy \\
&\quad + \alpha_3 \int_{\Omega^2} \delta^{-1} \left|G_\sigma(x,y,w_n) - G_\sigma(x,y,w)\right| \mu(x)\, dx dy
\end{aligned}$$

for some constant $C \ge 0$, because the function $F(\ell(\cdot))$ is differentiable and therefore Lipschitz on $[0; M_1]$, with M_1 given by (52). Now, given that

- $\|\mu_n - \mu\|_{L^1} \to 0$ (by Lemma A.2)
- $\left|G_\sigma(x,y,w_n) - G_\sigma(x,y,w)\right| \underset{n\infty}{\longrightarrow} 0$ for all $x, y \in \Omega$
- the function G_σ is bounded (for example, by $|\Omega|^{-1} e^{\frac{M_1 + \|c\|_\infty}{\sigma}}$)

The right term then goes to zero as n goes to infinity and is independent of z. Therefore,

$$\|f_{w_n,E} - f_{w,E}\|_{L^\infty} \to 0.$$

(2) Let us denote $f_n := f_{w,E_n}$, $f := f_{w,E}$, $\mu_n := \mu(w,E_n)$ and $\mu := \mu(w,E)$. Let $z \in \overline{\Omega}$. We have

$$|f_n(z) - f(z)| \le \alpha_3 \int_{\Omega^2} \delta^{-1} G_\sigma(x,y,w) \left|\mu_n(x) - \mu(x)\right| dx dy$$

The function G_σ is bounded and $\|\mu_n - \mu\|_{L^1} \to 0$ by Lemma A.2. Therefore, $\|f_{w,E_n} - f_{w,E}\|_{L^\infty} \to 0$.

Remark. We could simplify the proof by reasoning by composition. However, we opted for the current proof method because we needed Lipschitz estimates to prove the uniqueness result in Section 5. □

Lemma A.5. *Let $f \in L^\infty(\Omega)$, and $u_f \in H_0^1(\Omega)$ be the unique solution to the following equation*

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = f(z), & \forall z \in \Omega \\ u(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

There exists a constant $C(\Omega)$, depending only on Ω, such that

$$\|u_f\|_{L^2} \le C(\Omega)\|f\|_{L^\infty}$$

Proof. As u_f is a weak solution, we have, for any $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u_f \cdot \nabla v + \int_\Omega (\mathbf{V}\cdot\nabla u)\, v + \lambda \int_\Omega u_f\, v = \int_\Omega f v$$

Now, with $v = u_f$ we have, because λ is positive, and using Hölder inequality

$$\|\nabla u_f\|_{L^2}^2 \le \|f\|_{L^2}\, \|u_f\|_{L^2}$$

By Poincaré inequality

$$\|u_f\|_{L^2} \le \frac{\pi}{\mathrm{diam}(\Omega)} \|\nabla u_f\|_{L^2}$$

Then

$$\|u_f\|_{L^2} \le \frac{\pi}{\mathrm{diam}(\Omega)} \|f\|_{L^2} \le \frac{\pi|\Omega|^{1/2}}{\mathrm{diam}(\Omega)} \|f\|_{L^\infty} \quad \square$$

Corollary A.1. *Let (f_n) be a sequence in L^∞. If $\|f_n - f\|_{L^\infty} \to 0$ for some $f \in L^\infty$, then $\|u_{f_n} - u_f\|_{L^2} \to 0$ in $(K_2, \|\cdot\|_{L^2})$, where u_{f_n} is the unique solution to*

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = f_n(z), & \forall z \in \Omega \\ u(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

Proof. For every $n \in \mathbb{N}$, $u_{f_n} - u_f$ is solution to

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = (f_n - f)(z), & \forall z \in \Omega \\ u(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

By Lemma A.5, we have $C(\Omega)$ such that

$$\|u_{f_n} - u_f\|_{L^2} \le C(\Omega)\, \|f_n - f\|_{L^\infty}$$

which gives the desired convergence. □

We are now able to prove continuity of $\mathcal{Y}_2$ on $(K_2, \|\cdot\|_{L^2})$.

If (w_n) converges to w in $(K_1, \|\cdot\|_{L^\infty})$, by Lemma A.4, $f_{w_n,E}$ uniformly converges to $f_{w,E}$. Then, by Corollary A.1, $(\mathcal{Y}_2(w_n,E))$ goes to $\mathcal{Y}_2(w,E)$ in $L^2(\Omega)$.

Similarly, if (E_n) converges to E in $(K_2, \|\cdot\|_{L^\infty})$, by Lemma A.4, f_{w,E_n} uniformly converges to $f_{w,E}$. Then, by Corollary A.1, $(\mathcal{Y}_2(w,E_n))$ goes to $\mathcal{Y}_2(w,E)$ in $L^2(\Omega)$.

Proof of Proposition 5.1

For the existence part, the reasoning closely follows the continuous case. First, we obtain an explicit formula for the equilibrium distribution of residents. Indeed, if $(w,Q,E,\mu) \in (0,+\infty)^N \times C(\Omega,\mathbb{R}_+^*) \times \left(H_0^1(\Omega) \cap C(\Omega,\mathbb{R}_+^*)\right) \times \mathcal{P}_c(\Omega)$ is an equilibrium, then

$$\mu(x) = \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}\, \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}\, \psi(x)}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\, \tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\, \psi(y)\, dy}, \quad \forall x \in \overline{\Omega}.$$

The proof is very similar to that of Lemma 8.4 in Petit (2022).

By Lemma 8.2 in Petit (2022), w belongs to a convex and compact subset K_1 of $(0,+\infty)^N$, that only depends on the $(L_i)_{i=1,\dots,N}$, the $(c_i)_{i=1,\dots,N}$ and N.

Moreover, our proof of Proposition 4.3 can be easily adapted to show that E belongs to a convex and compact subset K_2 of $L^2(\Omega)$ that depends only on Ω, λ and N.

Let us define the map $\mathcal{Y} : K_1 \times K_2 \to K_1 \times K_2$ by the following construction:

(1) To any $(w,E) \in K_1 \times K_2$, we associate the probability $\mu(w,E)$ on Ω with density

$$\Omega \ni x \mapsto \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}\, \tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\, \tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\, dy} \tag{59}$$

with respect to the Lebesgue measure,

(2) We define $\mathcal{Y}_1(w,E)$ as the unique minimizer of

$$\min_{z\in(0,+\infty)^N} \left\{ \phi_{\mu(w,E)}(z) - \sum_{i=1}^N \int_{s=\varepsilon}^{z_i} L_i(s)ds \right\}.$$

(3) We define $\mathcal{Y}_2(w,E)$ as the unique solution of

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = f_{w,E}(z), & \forall z \in \Omega, \\ u(s) = 0, \quad \forall s \in \partial\Omega. \end{cases}$$

The map $\mathcal{Y}$ is well-defined, in particular, the existence of a unique minimizer in (2) is ensured by Lemma 8.2 in Petit (2022), and the existence of a unique solution to the PDE in (3) is guaranteed by our Proposition 4.3. From the proof of Theorem 8.2 in Petit (2022), the function $\mathcal{Y}_1$ is continuous with respect to w. It is also continuous with respect to E because if (E_n) converges to E in $(K_2, \|\cdot\|_{L^2})$, then by Lemma A.2 $(\mu(w,E_n))$ converges to $\mu(w,E)$ in $L^1(\Omega)$, and therefore weakly converges to the same limit. Then, by Lemma 8.3 in Petit (2022), $(\mathcal{Y}_1(w,E_n))$ uniformly converges to $\mathcal{Y}_1(w,E)$. The proof of the continuity of $\mathcal{Y}_2$ with respect to w and E can be very easily derived from the one given in the continuous case. Thus, $\mathcal{Y}$ is a continuous map from the convex and compact subset $K_1 \times K_2$ into itself. By Schauder's fixed-point theorem, it admits at least one fixed-point, which is an equilibrium in the sense of Definition 5.1. This concludes on the existence part.

Regarding the uniqueness of the solution, we first need to prove the following Lemma.

Lemma A.6. *There exists $\theta_0 > 0$, depending only on the $(L_i)_{i=1,\dots,N}$ and on w_0, such that for any $\theta \in [0;\theta_0]$, the equilibrium is unique if and only if the non-linear PDE*

$$\begin{cases} -\Delta E(z) + \mathbf{V}(z)\cdot\nabla E(z) + \lambda E(z) = f_{\mu(E),w(E)}(z), & \forall z \in \Omega \\ E(s) = 0, \quad \forall s \in \partial\Omega \end{cases} \tag{60}$$

admits a unique solution, where $(w(E), Q(E), \mu(E)) \in (0,+\infty)^N \times C(\Omega,\mathbb{R}_+^*) \times \mathcal{P}_c(\Omega)$ is defined as the unique solution of the system formed by Eqs. (32), (33) and (35) for any given $E \in H_0^1(\Omega)\cap C(\Omega,\mathbb{R}_+^*)$.

Proof. We know, from Theorem 8.3 in Petit (2022), that there exists $\theta_0 > 0$, depending only on the $(L_i)_{i=1,\dots,N}$ and on w_0, such that for every $\theta \in [0;\theta_0]$ and any given $E \in K_2$, the system formed by Eqs. (32), (33) and (35) admits a unique solution $(w(E), Q(E), \mu(E)) \in K_1 \times C(\Omega,\mathbb{R}_+^*)\times \mathcal{P}_c(\Omega)$.

Our claim is that, for $\theta \in [0;\theta_0]$, $(w_1, Q_1, E_1, \mu_1) \in (0,+\infty)^N \times C(\Omega,\mathbb{R}_+^*) \times \left(H_0^1(\Omega)\cap C(\Omega,\mathbb{R}_+^*)\right) \times \mathcal{P}_c(\Omega)$ is an equilibrium if and only if:

- $w_1 = w(E_1)$, $Q_1 = Q(E_1)$, $\mu_1 = \mu(E_1)$,
- and E_1 solves

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = f_{\mu(u),w(u)}(z), & \forall z \in \Omega \\ u(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

If (w_1, Q_1, E_1, μ_1) is an equilibrium, (w_1, Q_1, μ_1) is solution to (32)–(33)–(35) and consequently $w_1 = w(E_1)$, $Q_1 = Q(E_1)$ and $\mu_1 = \mu(E_1)$. Therefore, by Eq. (34),

$$\begin{cases} -\Delta E_1(z) + \mathbf{V}(z)\cdot\nabla E_1(z) + \lambda E_1(z) = f_{\mu(E_1),w(E_1)}(z), & \forall z \in \Omega \\ E_1(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

Reciprocally, it is easy to check that if $E_1 \in H_0^1(\Omega)\cap C(\Omega,\mathbb{R}_+^*)$ verifies

$$\begin{cases} -\Delta E_1(z) + \mathbf{V}(z)\cdot\nabla E_1(z) + \lambda E_1(z) = f_{\mu(E_1),w(E_1)}(z), & \forall z \in \Omega \\ E_1(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

and $w_1 = w(E_1)$, $Q_1 = Q(E_1)$, $\mu_1 = \mu(E_1)$, then (w_1, Q_1, E_1, μ_1) is an equilibrium.

Thus, there is a unique equilibrium if and only if the PDE

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = f_{\mu(u),w(u)}(z), & \forall z \in \Omega \\ u(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

admits a unique solution. □

Let $\theta \in [0,\theta_0]$, with θ_0 given by Lemma A.6. To prove uniqueness, we thus have to prove that Eq. (60) has a unique solution.

Let us consider the map $\mathcal{F}$ that associates, for any $E \in K_2$, the unique solution $u \in K_2$ to

$$\begin{cases} -\Delta u(z) + \mathbf{V}(z)\cdot\nabla u(z) + \lambda u(z) = f_{\mu(E),w(E)}(z), & \forall z \in \Omega \\ u(s) = 0, \quad \forall s \in \partial\Omega \end{cases}$$

We aim to prove that $\mathcal{F}$ is a contraction mapping. Let $\overline{\alpha} := \max\{\alpha_1,\alpha_2,\alpha_3\}$. Recall that $(\alpha_1,\alpha_2,\alpha_3)$ are the coefficients of the pollution source term $f_{\mu,w}$ given by Eq. (31). By Lemma A.5, there exists a constant C, independent of $(\alpha_1,\alpha_2,\alpha_3)$, such that for all $E_1, E_2 \in K_2$,

$$\|\mathcal{F}(E_1) - \mathcal{F}(E_2)\|_{L^2} \le C\|f_{w(E_1),\mu(E_1)} - f_{w(E_2),\mu(E_2)}\|_{L^\infty} \tag{61}$$

Now, for all $E_1, E_2 \in K_2$

$$\begin{aligned}\|f_{w(E_1),\mu(E_1)} - f_{w(E_2),\mu(E_2)}\|_{L^\infty} \le{}& \|f_{w(E_1),\mu(E_1)} - f_{w(E_2),\mu(E_1)}\|_{L^\infty} \\ &+ \|f_{w(E_2),\mu(E_1)} - f_{w(E_2),\mu(E_2)}\|_{L^\infty}\end{aligned} \tag{62}$$

By Lemma A.4 (1),

$$\begin{aligned}\|f_{w(E_1),\mu(E_1)} - f_{w(E_2),\mu(E_1)}\|_{L^\infty} \le{}& \overline{\alpha}C_1\|\mu(E_1) - \mu(E_2)\|_{L^1} \\ &+ \overline{\alpha}C_2\|w(E_1) - w(E_2)\|_{L^\infty}\end{aligned} \tag{63}$$

for some positive constants C_1 and C_2 independent of $(\alpha_1,\alpha_2,\alpha_3)$.

By definition, $\mu(E) = \tilde{\mu}(\cdot, w(E), E)$. Thus

$$\begin{aligned}\|\mu(E_1) - \mu(E_2)\|_{L^1} \le{}& \|\tilde{\mu}(\cdot, w(E_1), E_1) - \tilde{\mu}(\cdot, w(E_1), E_2)\|_{L^1} \\ &+ \|\tilde{\mu}(\cdot, w(E_1), E_2) - \tilde{\mu}(\cdot, w(E_2), E_2)\|_{L^1}\end{aligned} \tag{64}$$

First, by Lemma A.2 (2), and given that K_2 is bounded, we have $C_3 > 0$ (independent of $(\alpha_1,\alpha_2,\alpha_3)$) such that

$$\|\tilde{\mu}(\cdot, w(E_1), E_1) - \tilde{\mu}(\cdot, w(E_1), E_2)\|_{L^1} \le C_3\|E_1 - E_2\|_{L^2}. \tag{65}$$

Second, given that $w(E_1)$ and $w(E_2)$ belong to a compact subset K_1 (see Petit (2022), Lemma 8.2), and that for any $E \in K_2$ the map $(0,+\infty)^N \ni w \mapsto \tilde{\mu}(\cdot, w, E) \in L^1(\Omega)$ is C^1, it is therefore Lipschitz on K_1 and we have $C_4 > 0$ (independent of $(\alpha_1,\alpha_2,\alpha_3)$) such that

$$\|\tilde{\mu}(\cdot, w(E_1), E_2) - \tilde{\mu}(\cdot, w(E_2), E_2)\|_{L^1} \le C_4\|w(E_1) - w(E_2)\|_{L^\infty}.$$

Now, let us prove the following Lemma, which shows that the equilibrium wage map is locally Lipschitz with respect to the pollution distribution.

Lemma A.7. *For any $E \in H_0^1(\Omega)$, let $(w(E), Q(E), \mu(E)) \in (0,+\infty)^N \times C(\Omega,\mathbb{R}_+^*)\times\mathcal{P}_c(\Omega)$ be the unique solution of the system formed by Eqs. (32), (33) and (35). For all $\theta \in [0,\theta_0]$, the mapping $w : (H_0^1(\Omega), \|\cdot\|_{L^2}) \to ((0,+\infty)^N, \|\cdot\|_{L^\infty}), E \mapsto w(E)$ is locally Lipschitz, thus Lipschitz on the compact K_2.*

Proof. For $E \in H_0^1(\Omega)$ and $w \in (0,+\infty)^N$, let $g(w,E) := \Psi(w,E) - L(w)$, where:

- $L(w) = \left(L_i(w_i)\right)_{i\in\{1,\dots,N\}}$
- $\Psi(w,E) = \left(\int_\Omega G_{\sigma,i}(x,w)\tilde{\mu}(x,w,E)dx\right)_{i\in\{1,\dots,N\}}$
- $\tilde{\mu}(x,w,E) = \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}\tilde{E}(x)^{-\frac{\gamma}{1-\theta}}}{\int_\Omega R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\tilde{E}(y)^{-\frac{\gamma}{1-\theta}}dy}$, $\forall x \in \overline{\Omega}$

First, for any $E \in H_0^1(\Omega)$, $g(w(E),E) = 0$. Besides, the map g is of class C^1 on $(0,+\infty)^N \times H_0^1(\Omega)$. We have, for any $(w,E) \in (0,+\infty)^N \times H_0^1(\Omega)$:

- $D_w\Psi(w,E)(v) = \int_\Omega D_wG(x,w)\cdot v\,\tilde{\mu}(x,w,E)dx$, $\forall v \in (0,+\infty)^N$
- $D_E\Psi(w,E)(h) = \int_\Omega G(x,w)\frac{\tilde{\mu}(x,w,E)}{E(x)}h(x)dx$, $\forall h \in H_0^1(\Omega)$

Following Petit (2022), Theorem 8.3, for all $\theta \in [0,\theta_0]$, the matrix $D_wG(x,w)$ is invertible. Thus, by the implicit function theorem, for any $E \in H_0^1(\Omega)$, there exists $\phi_E \in C^1(H_0^1(\Omega),(0,+\infty)^N)$, $\mathcal{W}\times\mathcal{E}$ an open neighbourhood of $(w(E),E)$ such that, for all $(v,h) \in \mathcal{W}\times\mathcal{E}$ with $g(v,h) = 0$, we have $v = \phi(h)$.

This implies that the map $w : H_0^1(\Omega) \ni E \mapsto w(E) \in (0,+\infty)^N$ is locally Lipschitz, thus globally Lipschitz on the compact K_2. □

By Lemma A.7, the mapping $w : (K_2, \|\cdot\|_{L^2}) \to ((0,+\infty)^N, \|\cdot\|_{L^\infty}), E \mapsto w(E)$ is Lipschitz, thus we have $C_5 > 0$ and $C_6 > 0$ (independent of $(\alpha_1,\alpha_2,\alpha_3)$) such that

$$\|w(E_1) - w(E_2)\|_{L^\infty} \le C_5\|E_1 - E_2\|_{L^2} \tag{66}$$

and

$$\|\tilde{\mu}(\cdot, w(E_1), E_2) - \tilde{\mu}(\cdot, w(E_2), E_2)\|_{L^1} \le C_6\|E_1 - E_2\|_{L^2}. \tag{67}$$

Similarly, there exists $C_7 > 0$, independent of $(\alpha_1,\alpha_2,\alpha_3)$, such that

$$\|f_{w(E_2),\mu(E_1)} - f_{w(E_2),\mu(E_2)}\|_{L^\infty} \le \overline{\alpha}C_8\|\mu(E_1) - \mu(E_2)\|_{L^1}. \tag{68}$$

Combining inequalities (62), (63), (64), (65), (66), (67) and (68) yields the existence of $C_8 > 0$, independent of $(\alpha_1,\alpha_2,\alpha_3)$, such that

$$\|f_{w(E_1),\mu(E_1)} - f_{w(E_2),\mu(E_2)}\|_{L^\infty} \le \overline{\alpha}C_8\|E_1 - E_2\|_{L^2}.$$

As a consequence, if $\overline{\alpha}$ is small enough, then $\mathcal{F}$ is a contraction by inequality (61). Therefore Eq. (60) has a unique fixed-point. Thus, by Lemma A.6, there is a unique equilibrium.

Proof of Proposition 5.2

The proof is an adaptation of the proof of Achdou et al. (2023), Proposition 3.4. For every $\sigma \in (0,1)$, there exists $w_\sigma \in K_1$ and $E_\sigma \in K_2$, where K_1 is a compact subset of $(0,+\infty)^N$ and K_2 a compact subset of $L^2(\Omega)$ (both being independent of $\sigma \in (0,1)$), such that

$$\begin{aligned} \int_\Omega G_{\sigma,i}(x,w_\sigma)d\mu_\sigma(x) &= L_i(w_{\sigma,i}), \quad \forall i \in \{1,\dots,N\} \\ -\Delta E_\sigma(z) + \mathbf{V}(z)\cdot\nabla E_\sigma(z) + \lambda E_\sigma(z) &= f_{E_\sigma,w_\sigma}(z), \quad \forall z \in \Omega \\ E_\sigma(s) &= 0, \quad \forall s \in \partial\Omega \end{aligned}$$

where

$$\mu_\sigma(x) = \frac{R_\sigma(x,w_\sigma)^{\frac{\theta}{1-\theta}}\tilde{E}_\sigma(x)^{-\frac{\gamma}{1-\theta}}\psi(x)}{\int_\Omega R_\sigma(y,w_\sigma)^{\frac{\theta}{1-\theta}}\tilde{E}_\sigma(y)^{-\frac{\gamma}{1-\theta}}\psi(y)\,dy}, \quad \forall x \in \overline{\Omega}.$$

Up to a subsequence, we may assume that (w_σ, E_σ) converges to some $(w,E) \in K_1 \times K_2$ as σ goes to 0. By the proof of Proposition 5.1, E solves

$$\begin{aligned} -\Delta E(z) + \mathbf{V}(z)\cdot\nabla E(z) + \lambda E(z) &= f_{E,w}(z), \quad \forall z \in \Omega \\ E(s) &= 0, \quad \forall s \in \partial\Omega \end{aligned}$$

and μ_σ converges to $\mu \in \mathcal{P}_c(\overline{\Omega})$ in $L^1(\Omega)$, where

$$\mu(x) = \frac{R(x,w)^{\frac{\theta}{1-\theta}}\tilde{E}(x)^{-\frac{\gamma}{1-\theta}}\psi(x)}{\int_\Omega R(y,w)^{\frac{\theta}{1-\theta}}\tilde{E}(y)^{-\frac{\gamma}{1-\theta}}\psi(y)\,dy}, \quad \forall x \in \overline{\Omega}.$$

Thus, E satisfies (34). By using exactly the same arguments as in Achdou et al. (2023), proof of Proposition 3.4, we can show that (w,μ) satisfies (36) and (37). Finally, letting $Q := (1-\theta)R(\cdot,w)\mu$, we have that Q solves (33).

Therefore, the quadruplet (w,Q,E,μ) satisfies conditions (33), (34), (35), (36) and (37), completed with $E = 0$ on $\partial\Omega$. This proves the existence of equilibria without idiosyncratic shocks.

Proof of Proposition 6.1

Condition (19) is equivalent to the mean-field equation

$$\int_\Omega \mathrm{U}(x,\mu(x))d\mu(x) = \sup_{m\in\mathcal{P}_c(\Omega)} \int_\Omega \mathrm{U}(x,\mu(x))dm(x),$$

where for every $x \in \overline{\Omega}$,

$$\mathrm{U}(x,\mu) = \begin{cases} \theta^\theta \frac{R_\sigma(x,w)^\theta \tilde{E}(x)^{-\gamma}}{\mu(x)^{1-\theta}}, & \text{if } \mu \in \mathcal{P}_c(\overline{\Omega}), \\ -\infty & \text{otherwise,} \end{cases}$$

with the convention $1/0 = +\infty$. One can rewrite this equilibrium condition as follows:

$$\forall m \in \mathcal{P}(\overline{\Omega}), \quad \int_\Omega \mathrm{U}(x,\mu)d(m-\mu)(x) \le 0.$$

We recognize the first order condition of the following maximization problem:

$$\sup_{\mu\in\mathcal{P}(\overline{\Omega})} \int_\Omega \mathcal{U}(x,\mu)dx, \tag{69}$$

where $\mathcal{U}$ is the potential of the game. In this setting, U represents the derivative (in the sense of measures) of $\mathcal{U}$, defined by

$$\mathcal{U}(x,\mu) = \begin{cases} \theta^{\theta-1}\left(R_\sigma(x,w)\mu(x)\right)^\theta \tilde{E}(x)^{-\gamma}, & \text{if } \mu \in \mathcal{P}_c(\overline{\Omega}), \\ -\infty, & \text{otherwise.} \end{cases}$$

Since the utility function is a power function of the density of workers' residences, we observe that (69) is equivalent to

$$\sup_{\mu\in\mathcal{P}(\overline{\Omega})} \int_\Omega \mathrm{U}(x,\mu)d\mu(x).$$

Proof of Proposition 6.2

Applying the weak formulation of pollution dispersion (15), with a constant unit test function $v := 1$,[8] we get

$$\int_\Omega (\mathbf{V}\cdot\nabla E) + \lambda\int_\Omega E = \int_\Omega f_{\mu,w}.$$

But

$$\int_\Omega (\mathbf{V}\cdot\nabla E) = -\int_\Omega E(\nabla\cdot\mathbf{V}) + \int_{\partial\Omega} E(s)\mathbf{V}(s)\cdot\mathbf{n}\,ds = 0$$

by the divergence theorem and equations satisfied by E and $\mathbf{V}$: in particular, $\nabla\cdot V = 0$ and $E = 0$ on $\partial\Omega$. Therefore

$$\int_\Omega E = \lambda^{-1}\int_\Omega f_{\mu,w} = \lambda^{-1}\int_{\overline{\Omega}^2} |x-y|\,\mu(x)\,G_\sigma(x,y,w)\,dx\,dy$$

or, given that $m(x,y) := \mu(x)G_\sigma(x,y,w)$ is a probability density on Ω^2,

$$\int_\Omega E = \lambda^{-1}\mathbb{E}\left[|X-Y|\right]$$

where the couple (X,Y) follows the joint distribution of density $m(x,y)$.

Proof of Proposition 6.3

By Eq. (18), using μ as test function[9] we have

$$\lambda\int_\Omega E\mu = \int_\Omega \left(-\nabla E\cdot\nabla\mu + f_{\mu,w}\mu - \mathbf{V}\cdot\nabla E\mu\right)$$

with

$$\nabla\mu = \frac{\theta}{1-\theta}\frac{\nabla R_\sigma(\cdot,w)}{R_\sigma(\cdot,w)}\mu - \frac{\gamma}{1-\theta}\frac{\nabla E}{\tilde{E}}\mu. \tag{70}$$

Now, using the divergence theorem, boundary conditions on E, and incompressibility condition on $\mathbf{V}$, we obtain

$$-\int_\Omega \mathbf{V}\cdot\nabla E\,\mu = \int_\Omega E\left[\nabla\cdot(\mathbf{V}\mu)\right] = \int_\Omega E\,\nabla\mu\cdot\mathbf{V}$$

Thus

$$\begin{aligned} -\int_\Omega \mathbf{V}\cdot\nabla E\,\mu &= \frac{\theta}{1-\theta}\int_\Omega\left(\left[\mathbf{V}\cdot\frac{\nabla R_\sigma(\cdot,w)}{R_\sigma(\cdot,w)}\right]E\,\mu\right) - \frac{\gamma}{1-\theta}\int_\Omega \mathbf{V}\cdot\frac{\nabla E}{\tilde{E}}E\,\mu \\ &= \frac{\theta}{1-\theta}\int_\Omega\left(\left[\mathbf{V}\cdot\frac{\nabla R_\sigma(\cdot,w)}{R_\sigma(\cdot,w)}\right]E\,\mu\right) \\ &\quad - \frac{\gamma}{1-\theta}\int_\Omega \mathbf{V}\cdot\nabla E\,\mu + \frac{\gamma}{1-\theta}\int_\Omega \mathbf{V}\cdot\frac{\nabla E}{\tilde{E}}E_0\,\mu \end{aligned}$$

which yields

$$-\int_\Omega \mathbf{V}\cdot\nabla E\,\mu = \frac{\theta}{1-\theta-\gamma}\int_\Omega\left(\mathbf{V}\cdot\frac{\nabla R_\sigma(\cdot,w)}{R_\sigma(\cdot,w)}\right)E\,\mu + \frac{\gamma}{1-\theta-\gamma}\int_\Omega \mathbf{V}\cdot\frac{\nabla E}{\tilde{E}}\mu$$

because $E_0 = 1$. Now, by using the divergence theorem and formula (70), we obtain

$$\begin{aligned} \int_\Omega \mathbf{V}\cdot\frac{\nabla E}{\tilde{E}}\mu &= \int_\Omega \mathbf{V}\cdot\nabla(\ln(\tilde{E}))\,\mu \\ &= -\int_\Omega (\mathbf{V}\cdot\nabla\mu)\ln(\tilde{E}) + \int_{\partial\Omega}\ln(E_0)\mathbf{V}\mu \\ &= -\frac{\theta}{1-\theta}\int_\Omega\left(\left[\mathbf{V}\cdot\frac{\nabla R_\sigma(\cdot,w)}{R_\sigma(\cdot,w)}\right]\ln(\tilde{E})\,\mu\right) \\ &\quad + \frac{\gamma}{1-\theta}\int_\Omega \mathbf{V}\cdot\nabla(\ln(\tilde{E}))\ln(\tilde{E})\,\mu \\ &= -\frac{\theta}{1-\theta}\int_\Omega\left(\left[\mathbf{V}\cdot\frac{\nabla R_\sigma(\cdot,w)}{R_\sigma(\cdot,w)}\right]\ln(\tilde{E})\,\mu\right) \end{aligned}$$

[8] Note that we cannot formally use this test function since it is not equal to zero on $\partial\Omega$. However, it is possible to consider a sequence of $H_0^1(\Omega)$ converging to v, and apply the Lebesgue dominated convergence theorem to get the following equation.

[9] Same comment as the previous one.

$$+\frac{\gamma}{1-\theta}\int_{\Omega}\mathbf{V}\cdot\frac{\nabla E}{\tilde{E}}\ln(\tilde{E})\,\mu$$

We can iterate this calculation by integrating the second integral by parts, and show by mathematical induction that at the equilibrium

$$\int_{\Omega}\mathbf{V}\cdot\frac{\nabla E}{\tilde{E}}\,\mu=-\sum_{n=1}^{+\infty}\frac{\theta}{1-\theta}\left(\frac{\gamma}{1-\theta}\right)^{n}\int_{\Omega}\left(\mathbf{V}\cdot\frac{\nabla R_{\sigma}(\cdot,w)}{R_{\sigma}(\cdot,w)}\right)\frac{\ln(\tilde{E})^{n}}{n!}\,\mu$$

Hence, by dominated convergence

$$\begin{aligned}\int_{\Omega}\mathbf{V}\cdot\frac{\nabla E}{\tilde{E}}\,\mu&=-\frac{\theta}{1-\theta}\int_{\Omega}\left(\mathbf{V}\cdot\frac{\nabla R_{\sigma}(\cdot,w)}{R_{\sigma}(\cdot,w)}\right)\sum_{n=1}^{+\infty}\left(\frac{\gamma}{1-\theta}\right)^{n}\frac{\ln(\tilde{E})^{n}}{n!}\,\mu\\&=-\frac{\theta}{1-\theta}\int_{\Omega}\left(\mathbf{V}\cdot\frac{\nabla R_{\sigma}(\cdot,w)}{R_{\sigma}(\cdot,w)}\right)\left(\tilde{E}^{\frac{\gamma}{1-\theta}}-1\right)\mu\end{aligned}$$

which gives the desired result.

References

Achdou, Yves, Carlier, Guillaume, Petit, Quentin, Tonon, Daniela, 2023. A simple city equilibrium model with an application to teleworking. Appl. Math. Optim. (ISSN: 0095-4616) 88 (2).

Achdou, Yves, Han, Jiequn, Lasry, Jean-Michel, Lions, Pierre-Louis, Moll, Benjamin, 2021. Income and wealth distribution in macroeconomics: A continuous-time approach. Rev. Econ. Stud. (ISSN: 0034-6527) 89 (1), 45–86.

AirParif, 2019. Emissions de polluants atmosphériques et de gaz à effet de serre - Bilan île-de-France - Année 2019.

Allen, Treb, Arkolakis, Costas, Li, Xiangliang, 2020. On the Equilibrium Properties of Network Models with Heterogeneous Agents. Working Paper 27837, National Bureau of Economic Research.

Alvarez, Fernando, Lucas, Robert E., 2007. General equilibrium analysis of the Eaton–Kortum model of international trade. J. Monetary Econ. (ISSN: 0304-3932) 54 (6), 1726–1768. http://dx.doi.org/10.1016/j.jmoneco.2006.07.006.

Anas, Alex, 1990. Taste heterogeneity and urban spatial structure: The logit model and monocentric theory reconciled. J. Urban Econ. (ISSN: 0094-1190) 28 (3), 318–335.

Arnott, Richard, Hochman, Oded, Rausser, Gordon C., 2008. Pollution and land use: Optimum and decentralization. J. Urban Econ. (ISSN: 0094-1190) 64 (2), 390–407.

Bahlali, Meissam L., Dupont, Eric, Carissimo, Bertrand, 2019. Atmospheric dispersion using a Lagrangian stochastic approach: Application to an idealized urban area under neutral and stable meteorological conditions. J. Wind Eng. Ind. Aerodyn. (ISSN: 0167-6105) 193, 103976.

Borck, Rainald, 2019. Public transport and urban pollution. Reg. Sci. Urban Econ. 77, 356–366.

Boucekkine, Raouf, Fabbri, Giorgio, Federico, Salvatore, Gozzi, Fausto, 2021. From firm to global-level pollution control: The case of transboundary pollution. European J. Oper. Res. (ISSN: 0377-2217) 290 (1), 331–345.

Boucekkine, Raouf, Fabbri, Giorgio, Federico, Salvatore, Gozzi, Fausto, 2022. A dynamic theory of spatial externalities. Games Econom. Behav. (ISSN: 0899-8256) 132, 133–165.

Camacho, Carmen, Pérez-Barahona, Agustín, 2015. Land use dynamics and the environment. J. Econom. Dynam. Control (ISSN: 0165-1889) 52, 96–118.

Diamond, Rebecca, 2016. The determinants and welfare implications of US workers' diverging location choices by skill: 1980–2000. Amer. Econ. Rev. 106 (3), 479–524.

Fontenla, Matias, Ben Goodwin, M., Gonzalez, Fidel, 2019. Pollution and the choice of where to work and live within Mexico City. Latin Am. Econ. Rev. (ISSN: 2196-436X) 28 (1), 11.

Fujita, Masahisa, Ogawa, Hideaki, 1982. Multiple equilibria and structural transition of non-monocentric urban configurations. Reg. Sci. Urban Econ. (ISSN: 0166-0462) 12 (2), 161–196.

Heblich, Stephan, Trew, Alex William, Zylberberg, Yanos, 2021. East-side story: Historical pollution and persistent neighborhood sorting. J. Polit. Econ. 129, 1508–1552.

Kyriakopoulou, Efthymia, Picard, Pierre M., 2021. On the design of sustainable cities: Local traffic pollution and urban structure. J. Environ. Econ. Manag. (ISSN: 0095-0696) 107, 102443.

La Torre, Davide, Liuzzi, Danilo, Marsiglio, Simone, 2021. Transboundary pollution externalities: Think globally, act locally? J. Math. Econ. (ISSN: 0304-4068) 96, 102511.

Le Dret, Hervé, 2013. Équations Aux Dérivées Partielles Elliptiques Non Linéaires. Springer.

Lucas, D.H., 1958. The atmospheric pollution of cities. Int. J. Air Pollut. (ISSN: 0304-4068) 71–86.

Lucas, Robert E., Rossi-Hansberg, Esteban, 2002. On the internal structure of cities. Econometrica 70 (4), 1445–1476.

Petit, Quentin, 2022. Mean Field Games and Optimal Transport in Urban Modelling (Ph.D. thesis). Paris Dauphine University.

Powell, Michael J.D., 1970. A hybrid method for nonlinear equations. In: Numerical Methods for Nonlinear Algebraic Equations. pp. 87–114.

Regnier, Camille, Legras, Sophie, 2018. Urban structure and environmental externalities. Environ. Resource Econ. 70 (1), 31–52.

Schindler, Mirjam, Caruso, Geoffrey, Picard, Pierre, 2017. Equilibrium and first-best city with endogenous exposure to local air pollution from traffic. Reg. Sci. Urban Econ. (ISSN: 0166-0462) 62, 12–23.

Smith, V. Kerry, Huang, Ju Chin, 1993. Hedonic models and air pollution: Twenty-five years and counting. Environ. Resource Econ. (ISSN: 1573-1502) 3 (4), 381–394.

Sportisse, Bruno, 2009. Fundamentals in Air Pollution. Springer.

Tchepel, O., Costa, A.M., Martins, H., Ferreira, J., Monteiro, A., Miranda, A.I., Borrego, C., 2010. Determination of background concentrations for air quality models using spectral analysis and filtering of monitoring data. Atmos. Environ. 44 (1), 106–114.

Wegener, Michael, 2019. Land-use transport interaction models. In: Fischer, Manfred M., Nijkamp, Peter (Eds.), Handbook of Regional Science. Springer Berlin Heidelberg, Berlin, Heidelberg, ISBN: 978-3-642-36203-3, pp. 1–18.