A mean field model for the interactions between firms on the markets of their inputs

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Motivation

Focusing on one sector of activity, firms have interactions between them

1. Through the competition when they sell their production.
2. Through the competition when they buy factors of production such as labour or workspace for instance.
3. Through contacts which create externalities.

   ...

$\Rightarrow$ We focus on modelling the second point.

  $\rightarrow$ It allows us to link several markets.

Some references

MFG theory:

- Lasry and Lions 2006–2007
- Huang, Malhamé and Caines 2006 – 2007
- Carmona and Delarue 2018
- Cardaliaguet, Delarue, Lasry and Lions 2019
- Lions 2006 – 2022
- ...

MFG price formation model:

- Trading: Lachapelle et al. 2016, Cardaliaguet and Lehalle 2018
- Energy market: Gomes and Saúde 2018–2021
- Exhaustible resources: Guéant et al. 2011
- Growth theory: Achdou et al. 2022
- ...

General assumptions and outputs of the model

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- There is one sector in the economy with a (very) large number of firms which are rational and indistinguishable

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The outputs are

- The unit prices of the production factors
- The distribution of the firms' capital
- The investment strategy of firms

The dynamics of a firm's capital

For a given firm, the dynamics of its capital $k(t)$ is given by

$$\frac{dk}{dt}(t) = F(k(t), \ell(t)) - w \cdot \ell(t) - \delta k(t) - c(t), \quad \forall t \geq 0$$

where

- $d \in \mathbb{N}^*$ is the number of factors of production or inputs

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- $\ell(t) \in \mathbb{R}^d$ stands for the quantities of inputs at time t
- $\delta \geq 0$ is the depreciation rate of capital
- $c(t) \geq 0$ stands for the consumption of the owner of the firm at time t

The optimal control problem

$$u(\kappa) = \sup_{c(t),\, \ell(t)} \int_0^{+\infty} U(c(t)) e^{-\rho t} dt$$

subject to

$$\left\{ (c, k, \ell) : \begin{array}{l} c, \ell \in L^1_{\text{loc}}(0, +\infty), \quad k \in W^{1,1}_{\text{loc}}(0, +\infty), \\ c \geq 0, \quad \ell \geq 0, \\ k \geq 0, \quad k(0) = \kappa, \end{array} \right.$$

where

- $\kappa > 0$ is the initial capital

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- $U : (0, +\infty) \to \mathbb{R}$ is a utility function

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where

- $\kappa > 0$ is the initial capital
- $U : (0, +\infty) \to \mathbb{R}$ is a utility function
- $\rho > 0$ is the discount rate
- we impose $k \geq 0$

Hamilton-Jacobi equation

Given $w \in (0, +\infty)^d$, the optimal control problem leads to the HJ equation

$$\rho u(k) = H\left(k, u'(k)\right)$$

with the state constraint boundary condition

$$D_q H(0, u'(0)) \geq 0$$

where $H : [0, +\infty) \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\begin{aligned} H(k,q) &= \sup_{c \geq 0,\ \ell \in [0,+\infty)^d} \left\{ U(c) + q\left(F(k,\ell) - w\ell - \delta k - c\right)\right\} \\ &= \sup_{c \geq 0} \left\{ U(c) - cq \right\} + f(k)q \end{aligned}$$

with $f(k) = \sup_{\ell \in [0,+\infty)^d} \left\{ F(k,\ell) - w\ell \right\} - \delta k$.

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with $f(k) = \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w\ell\} - \delta k$.

If u is smooth enough, the investment policy is $D_q H\left(k, u'(k)\right)$.

Continuity equation

Given the optimal policies of the firms, the distribution of capital solves the continuity equation

$$\frac{d}{dk}\left(D_q H\left(\cdot, \frac{\partial u}{\partial k}(\cdot)\right) m(\cdot)\right)(k) = \eta(k, u(k)) - \nu m(k), \qquad \forall k > 0$$

where

- $\nu > 0$ is the death rate of firms
- $\eta : (0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is a function which models the creation of new firms
  - The second variable takes into account the level of utility: external investors can decide to enter the game only if the level of utility is high enough.

Market clearing conditions

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The collection of unit prices $w \in (0, +\infty)^d$ must satisfy the law of the supply and demand, i.e.

$$S(w) = -\int_0^{+\infty} D_w f(k) dm(k)$$

where $S : [0, +\infty)^d \to [0, +\infty)^d$ is given and models the supply of factors of production.

The stationary system

$$\begin{aligned} \rho u(k) &= H\left(k, u'(k)\right) \\ \frac{d}{dk}\left(D_q H\left(\cdot, u'(\cdot)\right) m(\cdot)\right)(k) &= \eta(k, u(k)) - \nu m(k) \\ S(w) &= -\int_0^{+\infty} D_w f(k) dm(k) \end{aligned}$$

completed with the following conditions:

$$\begin{aligned} \text{''}\ D_q H(0, u'(0)) &\geq 0\ \text{''} \\ \int_0^{+\infty} dm(k) &= \frac{1}{\nu}\int_0^{+\infty} \eta(k, u(k)) dk \end{aligned}$$

Definition

An equilibrium is a triplet (u, m, w) solution to the system above where the HJ equation is satisfied in the viscosity sense and the continuity equation in the sense of distributions.

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We assume that $d = 1$, $F(k, \ell) = Ak^{\alpha}\ell^{1-\alpha}$, $U(c) = \ln(c)$.

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Finding the MFG equilibria boils down to solving

$$S(w) = \frac{1}{\nu - b(w)} \left(\frac{A(1-\alpha)}{w} \right)^{\frac{1}{\alpha}} \int_0^{+\infty} \kappa \eta(\kappa) d\kappa$$

where $b(w) = Cw^{-\frac{1-\alpha}{\alpha}} - \delta - \rho$.

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Theorem

If $w \mapsto S(w)$ is a continuous and non decreasing function, non identically 0 such that $S(0) = 0$ and if $\int_0^{\infty} \kappa \eta(\kappa) d\kappa < +\infty$, then there exists a unique solution of the MFG system.

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- if $\eta(\cdot)$ has a compact support then the right tail of the distribution $m(\cdot)$ decays like a power of k.
  This is known as Pareto's law in economics.

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- Using monotonicity properties of $H(\cdot, \cdot)$ and applying a shooting method, we prove the existence of solutions $u(\cdot) \in C^1(0, +\infty)$
- Concerning uniqueness, we can prove a verification theorem

Some properties the optimal investment strategy

We prove that there exists a critical value $\kappa^* > 0$ such that the optimal investment of a firm with capital k, namely $D_q H(k, u'(k))$, satisfies

$$\begin{aligned} D_q H\left(k, u'(k)\right) &> 0, \qquad \text{for} \quad 0 < k < \kappa^*, \\ D_q H\left(k, u'(k)\right) &< 0, \qquad \text{for} \quad \kappa^* < k < +\infty. \end{aligned}$$

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This is known as the golden rule of accumulation of capital in economics.

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Furthermore, there exists $\epsilon > 0$ and $M > 0$ such that

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$$|\kappa^* - k| \le \epsilon \Rightarrow \left|D_q H(k, u'(k))\right| \le M\,|\kappa^* - k|$$

$\Rightarrow$ the firms' capital takes an infinite time to reach κ^*

Existence of equilibria

The continuity equation admits a unique solution in the distributional sense (with no Dirac mass at κ^*) which belongs to $C^1((0,\kappa^*)\cup(\kappa^*,+\infty))$.

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Theorem (the decreasing return to scale case)

Under some technical assumptions, if

- *the supply function admits a convex potential, i.e. there exists a strictly convex function* $\Phi : (0,+\infty)\to\mathbb{R}$, C^1 *regular, such that* $D_w\Phi = S$,
- *and that there exist a continuous density* $\hat\eta$ *with compact support in* $\mathbb{R}_+^*$ *and* $\hat c \geq 1$ *satisfaying*

$$\frac{1}{\hat c}\hat\eta(k) \leq \eta(k,u) \leq \hat c\hat\eta(k), \quad \forall k\geq 0,\ \forall u\in\mathbb{R},$$

then there exists an equilibrium.

Existence of equilibria

Sketch of proof.

Define for every $w \in (0, +\infty)^d$,

$$T_\lambda(w) = \operatorname{argmin} \Big\{ \Phi(\cdot) + \int_{\mathbb{R}_+} (f(k,\cdot) + \delta k)\left((1-\lambda) d\hat{\eta}(k) + \lambda dm(k,w)\right) \Big\},$$

where $m(\cdot, w)$ is the distribution of capital.

1. Observe that if $w = T_1(w)$, then it defines an equilibrium (first order condition).
2. Observe that T_0 is a constant function
   $\Rightarrow$ given $\hat{w}$, $w_0 = T_0(\hat{w})$ satisfies $w = T_0(w)$.
3. Establish the continuity of T_λ

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where $m(\cdot, w)$ is the distribution of capital.

4. Establish a priori bounds independant of λ for the solutions of $w = T_\lambda(w)$.
5. Use Brouwer degree theory to conclude that there exists a solution to $w = T_1(w)$.

   $\Rightarrow$ There exists an equilibrium.

□

Simulations: linking the rental and labour markets

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$$F(k,(\ell_1,\ell_2)) = Ak^{\alpha}\ell_1^{\beta_1}\ell_2^{\beta_2} \quad \forall k \in \mathbb{R}_+,\ \forall(\ell_1,\ell_2) \in \mathbb{R}_+^2$$

- The utility function $U(\cdot) = \ln(\cdot)$
- The supply functions are of the form (logistic functions)

$$K/(1+e^{-r(w_i - w_0^i)}) \quad (i = 1, 2)$$

  where w_1 is the wages and w_2 the rental price
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⇒ We find an equilibrium using a numerical method similar to the one developed in Achdou et al. 2022.

Data from INSEE

Secteur d'activité	Nombre d'entreprises	Salariés (en EQTP) (1)	Chiffre d'affaires hors taxes (en millions d'euros)	Valeur ajoutée hors taxes (en millions d'euros)	Frais de personnel (en millions d'euros) (2)	Exportations (en millions d'euros)	Salariés / Nombre entreprises
Hébergement-restauration	**258 278**	**842 788**	**104 859**	**45 175**	**34 371**	**2 086**	**3.263104**
Hébergement	51 354	198 113	29 223	12 536	8 428	763	**3.857791**
Restauration	206 924	644 675	75 636	32 639	25 943	1 323	**3.115516**
Information et communication	**134 700**	**758 065**	**203 851**	**95 794**	**62 175**	**31 021**	**5.627803**
Édition, audiovisuel et diffusion	41 934	243 836	69 973	27 833	19 771	12 800	**5.814757**
Télécommunications	2 986	142 923	57 870	28 514	11 209	3 795	**47.864367**
Activités informatiques et services d'information	89 780	371 307	76 009	39 447	31 195	14 427	**4.135743**
Activités immobilières	**213 534**	**218 697**	**85 161**	**43 777**	**13 766**	**1 317**	**1.024179**
Activités scientifiques et techniques ; services administratifs et de soutien	**690 420**	**1 802 620**	**337 567**	**179 189**	**140 117**	**41 475**	**2.610904**
Activités scientifiques et techniques	5 875	38 313	7 556	2 156	3 079	3 525	**6.521362**
Activités juridiques, comptables, de gestion, architecture, ingénierie, contrôle et analyses	362 947	684 820	146 234	79 453	58 401	21 131	**1.886832**
Autres activités spécialisées, scientifiques et techniques	120 294	128 092	29 651	12 504	9 509	3 762	**1.064825**
Services administratifs et de soutien	201 304	951 395	154 126	85 075	69 128	13 057	**4.726160**
Autres activités de services	**365 522**	**337 112**	**55 519**	**23 619**	**15 899**	**2 013**	**0.922276**
Arts, spectacles et activités récréatives	144 033	121 092	30 426	12 182	7 419	1 101	**0.840724**
Autres activités de services	221 489	216 020	25 094	11 437	8 480	912	**0.975308**
Total	**1 662 454**	**3 959 281**	**786 958**	**387 554**	**266 328**	**77 913**	**2.381588**

Table: Characteristics of mainly market services by activity in 2018

Parameters in the simulation

Parameter	Test
d	2
α	0.21
(β_1, β_2)	(0.71,0.05)
δ	0.07
ν	0.1
ρ	0.1
A	$0.93.10^4$
$S_{labour}(w)$	$\frac{6.5}{1+\exp(2.10^{-4}(w-7.10^4))}$
$S_{workspace}(p)$	$\frac{100}{1+\exp(2.10^{-2}(p-300))}$
$\eta(k)$	$\frac{\nu}{\sqrt{2\pi}9.10^4} e^{-\frac{(k-3.10^5)^2}{2(9.10^4)^2}}$

Table: The parameters of the simulation.

Simulations: linking the office rental and labour markets

At the equilibrium, the wages is 8.13×10^4 Euro/year and the rental price is 391 Euro/(year.m^2).

Simulations: linking the office rental and labour markets

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	Test	INSEE
Annual prod. / Nb of firms (in 10^4 Euro)	67.4	66.4
Tot. payroll / Nb of firms (in 10^4 Euro)	47.9	47.1
Nb of Employees / Nb of firms	5.89	5.81
Annual prod. / Nb of Employees (in 10^4 Euro)	11.5	11.4
Tot. payroll / Nb of Employees (in 10^4 Euro)	8.13	8.11

Table: Comparative table

The distribution of capital

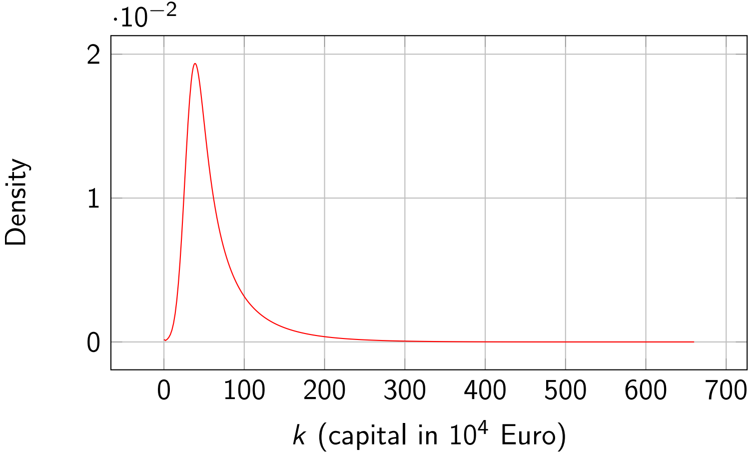

Figure: Distribution of capital m

The value function

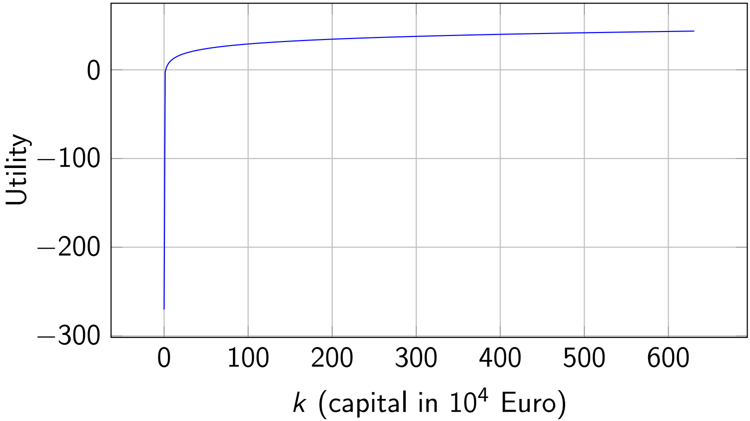

Figure: Value function u

Sensitivity with respect to the labour supply

We consider that

$$S_{labour}(w) = \lambda \frac{K}{1 + e^{-r(w-w_0)}}.$$

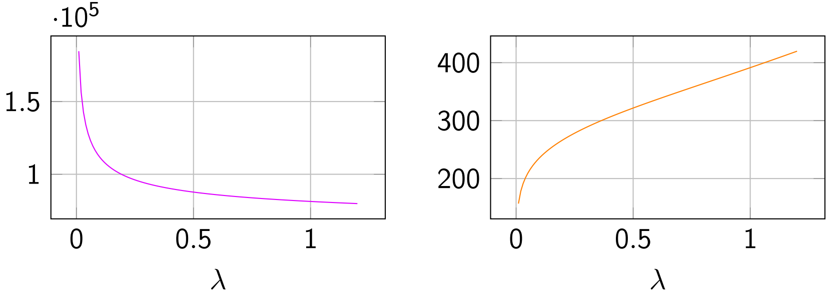

Figure: The wage level (pink) and the rental price (orange)

Conclusion (1/2)

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- For more details, see the preprint:

  A mean field model for the interactions between firms on the markets of their inputs

  and the PhD manuscript:

  Mean field games and optimal transport in urban modelling

Conclusion (2/2)

Open problems and perspectives

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1. Take into account interactions via controls

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1. Take into account interactions via controls
2. Introduce noise in the dynamics of capital
3. Extend the theory to evolutive models

Thank you for your attention.