A simple city equilibrium model with an application to teleworking

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Motivations

Workers decide the position of their home depending of several characteristics.

1. The distance between their home and their workplace.
2. The rental price.
3. Their revenue.
4. Their social activities proposed.
5. ...

$\Rightarrow$ We focus on modelling the three first points.

- $\rightarrow$ It allows us to link the labour and residential housing market.

Somes references

Non-atomic static games :

- Aumann 1964
- Carmona 2005
- ...

Urban spatial modelling :

- von Thünen 1826 ; Beckmann 1957 ; Alonso 1964, Mills 1972 and Muth 1969
- Fujita 1989 ; Lucas and Rossi-Hansberg 2002
- Carlier and Ekeland 2004–2007 ; Barilla et al. 2021
- ...

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The outputs are

- the collection of wages
- the rental price
- the distribution of the residences of workers depending of their workplace

Firms

We consider a city modelled by X a compact subset of $\mathbb{R}^d$

The firm $i \in \{1, ..., N\}$ is located at $y_i \in X$ and has a production function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$

If the wage paid by the firm i is denoted by w, its employment level is obtained by maximizing profit :

$$\pi_i(w) = \sup_{l \geq 0}\{f_i(l) - wl\}.$$

$\Rightarrow$ The envelope theorem yields that for every $w > 0$, firm i's labour demand is given by

$$L_i(w) = \operatorname{argmax}\{f_i(l) - wl\} = -\pi_i'(w).$$

Workers preferences

We assume that the utility of a given worker is

$$\mathcal{U}_\theta(R, Q) = \sup\left\{ C^\theta S^{1-\theta} \;:\; C + QS \leq R, C \geq 0, S \geq 0 \right\}$$

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$$C_\theta(R) = \theta R \qquad \text{and} \qquad S_\theta(R, Q) = (1 - \theta)\frac{R}{Q}$$

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Hence the utility of a worker is given by

$$\mathcal{U}_\theta(R, Q) = \theta^\theta (1-\theta)^{1-\theta} \frac{R}{Q^{1-\theta}}$$

Deducing density from revenues

Revenues $R(x)$, surface consumption $S(x)$ and rents $Q(x)$ are functions of the workers' residential location $x \in X$.

If we denote by $\mu(x)$ the probability density of the workers' residential distribution, μ and S are simply related by

$$\mu(x)S(x) = 1, \quad \forall x \in X.$$

$\rightarrow$ In particular the support of μ is the whole city X.

At equilibrium, utility should be constant which yields

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$$\mathcal{U}_\theta(R, Q) \propto 1 \Leftrightarrow Q \propto R^{\frac{1}{1-\theta}},$$

$\Rightarrow$ This relation, the fact that $S(x) = (1-\theta)\frac{R(x)}{Q(x)}$, and the market clearing conditions on the housing market yield

$$\mu(x) = \frac{R(x)^{\frac{\theta}{1-\theta}}}{\int_X R(y)^{\frac{\theta}{1-\theta}} dy}$$

Free-mobility of labour and commuting costs

For $i = 1, ..., N$, we denote by $c_i : X \to \mathbb{R}_+$ the transportation costs to reach the firm i.

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Then the revenue of a worker is

$$R(x, w) = \max_{i \in \{0,...,N\}} (w_i - c_i(x))$$

Labour supply

To facilitate the analysis, we replace $R(x, w)$ by

$$R_\sigma(x, w) = \sigma \ln \left(\sum_{i=0}^{N} e^{\frac{w_i - c_i(x)}{\sigma}} \right)$$

which can be justify by adding a random term to the wages.

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$\Rightarrow$ The probability to choose the firm i for a worker in the position $x \in X$ is given by the Gibbs distribution :

$$\frac{\partial R_\sigma}{\partial w_i}(x, w) = \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{k=0}^{N} e^{\frac{w_k - c_k(x)}{\sigma}}}$$

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$\Rightarrow$ For any $(w, \mu) \in (0, +\infty)^N \times \mathcal{P}(X)$, the labour supply for the firm i is

$$\int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x)$$

Definition of equilibria

We want to solve

$$\pi_i'(w_i) + \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu_w(x) = 0, \quad \forall i \in \{1, ..., N\},$$

where

$$\mu_w(x) = \frac{R(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R(y, w)^{\frac{\theta}{1-\theta}} dy}, \quad \forall x \in X.$$

Definition (equilibrium)

An equilibrium is a vector of wages $(w_1, ..., w_N) \in (0, +\infty)^N$ solution of the system above.

Main results

Theorem

We assume that for every $i \in \{1, ..., N\}$, the production function f_i is continuous on $\mathbb{R}_+$, C^1 on $\mathbb{R}_+^*$, and satisfies

$$f_i(0) \geq 0, \quad \lim_{l \to +\infty} f_i(l) = +\infty, \quad \lim_{l \to 0^+} f_i'(l) = +\infty, \quad \lim_{l \to +\infty} f_i'(l) = 0.$$

Existence : *There exists an equilibrium.*

Uniqueness : *If moreover, for every $i \in \{1, .., N\}$, f_i is $C^2(0, +\infty)$ and $f_i''(l) < 0$ for every $l \in \mathbb{R}_+^*$,*
then there is an explicit constant θ_0 such that for every $\theta \in [0, \theta_0]$ the equilibrium is unique.

Existence of equilibria (1/2)

Given a probability $\mu \in \mathcal{P}(X)$, the equilibrium condition on the labour market

$$\pi_i'(w_i) + \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x) = 0, \quad \forall i \in \{1, ..., N\},$$

is the first-order optimality condition equation for the convex minimization problem

$$\inf_{w \in \mathbb{R}_+^N} J_\mu(w) \text{ where } J_\mu(w) := \sum_{i=1}^{N} \pi_i(w_i) + \int_X R_\sigma(x, w) \mathrm{d}\mu(x).$$

Lemma

For every $\mu \in \mathcal{P}(X)$, the optimization problem admits a unique minimizer $w^*(\mu)$ and there exist constants $\underline{w}$ and $\overline{w}$ that do not depend on μ such that $0 < \underline{w} < \overline{w}$ and $w^*(\mu) \in [\underline{w}, \overline{w}]^N$.

Moreover $w^*(\mu)$ is the only solution of system above and the map $\mu \in \mathcal{P}(X) \mapsto w^*(\mu) \in [\underline{w}, \overline{w}]^N$ is weakly $*$ continuous.

Existence of equilibria (2/2)

We prove the existence of equilibria with a fixed-point strategy. Defining

$$\Phi(w) = w^*(\mu_w),$$

where

$$\mu_w(x) = \frac{R(x,w)^{\frac{\theta}{1-\theta}}}{\int_X R(y,w)^{\frac{\theta}{1-\theta}}\,dy}, \quad \forall x \in X.$$

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- $\Rightarrow$ Φ is a continuous self-map of $[\underline{w}, \overline{w}]^N$.
- $\Rightarrow$ Applying Brouwer fixed-point theorem, we deduce that there exists an equilibrium.

Uniqueness

Recalling the explicit formula for the distribution of workers, and defining

$$\mu_w(x,\theta) := \frac{R_\sigma(x,w)^{\frac{\theta}{1-\theta}}}{\int_X R_\sigma(y,w)^{\frac{\theta}{1-\theta}}\mathrm{d}y},\ \forall (x,w,\theta) \in X \times \mathbb{R}_+^N \times [0,1)$$

we see that finding an equilibrium amounts to solving

$$G(w,\theta) = 0,$$

where for every i, $G_i(w,\theta) = \pi_i'(w_i) + \int_X \frac{\partial R_\sigma}{\partial w_i}(x,w) d\mu_w(x,\theta)$. We observe that

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1. For $\theta = 0$, $\widetilde{\mu}_0 := \mu_w(x,0)$ does not depend on w and is the density of the uniform probability measure on X.

   $\Rightarrow$ There exists a unique equilibrium which is the unique minimizer of the strictly convex function $J_{\widetilde{\mu}_0}$.

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2. We can use the particular structure of the Jacobian of $G(\cdot, \theta)$ and apply the implicit function theorem to extend uniqueness up to $\theta_0 > 0$.

Numerical simulations with telework

We assume that

- $X = [-40, 40]$ and there are 3 working places in -25, 0 and 10
- The transportation cost is linear for the commuters and free for the telecommuters
- We assume that

$$f_i(\ell_1, \ell_2) = A_i(\ell_1^\gamma + B\ell_2^\gamma)^{\frac{\beta}{\gamma}},$$

with γ and β in $(0, 1)$.

$\Rightarrow$ Recall that for each working place the demand for labour is

$$L_i(w_1, w_2) = \text{argmax}\{f_i(\ell_1, \ell_2) - w_1\ell_1 - w_2\ell_2\} \in \mathbb{R}^2.$$

Parameters in the simulation

Parameter	Value
β	0.7
γ	0.9
A_i	20
w_0	12
σ	0.1
θ	0.7

Table – The parameters of the simulation.

Numerical simulations with telework in 2D

We run the same simulation with $X = [-10, 10]^2$

The workplaces are located in $(-7, 7)$, $(0, 0)$ and $(3, -3)$

Numerical simulations with telework in 2D

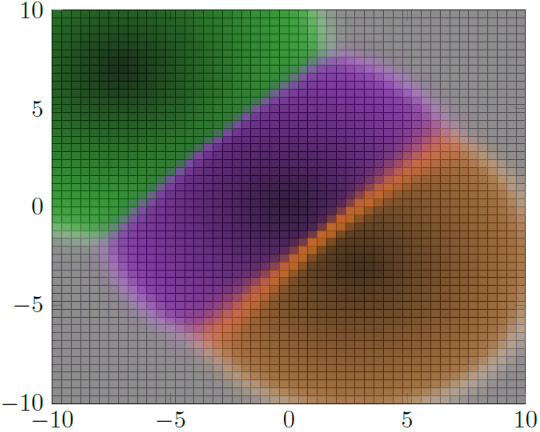

Figure – The distributions of workers for $B = 0$

Numerical simulations with telework in 2D

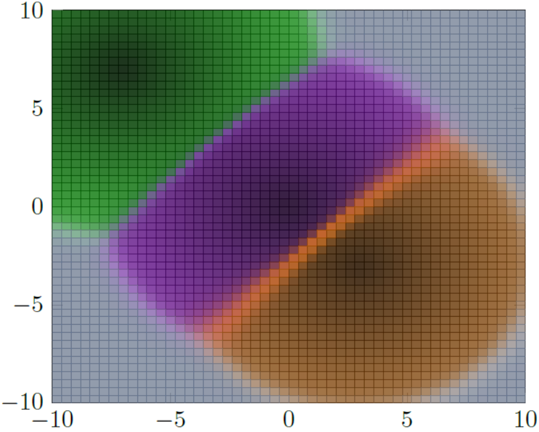

Figure – The distributions of workers for $B = 0.5$

Numerical simulations with telework in 2D

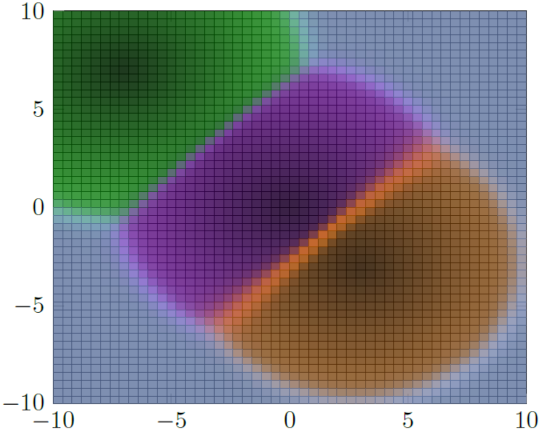

Figure – The distributions of workers for $B = 0.66$

Numerical simulations with telework in 2D

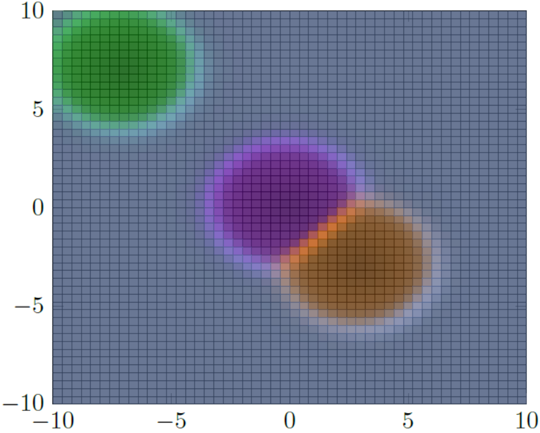

Figure – The distributions of workers for $B = 1$

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  3. A model with several types of workers

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- For more details, see the preprint :

  A simple city equilibrium model with an application to teleworking

  and the PhD manuscript :

  Mean field games and optimal transport in urban modelling

Thank you for your attention.